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Conditional Median as a Robust Solution Concept for Location Problems*

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Abstract

While making location decisions one intend to increase effects (reduce distances) for many the service recipients (clients). A conventional optimization approach to location problems considers only the optimality of locational decisions for specific clients data. Real-world applications inevitably involve errors and uncertainties in the operating conditions and thereby the resulting performance may be lower than expected. In particular, a distribution system design is very sensitive to the varying demands for goods and changed demands may deteriorate drastically the system efficiency when optimized for different demands structure. Several approaches have been developed to deal with uncertain or imprecise data. The approaches focused on the quality or on the variation (stability) of the solution for some data domains are considered robust. Frequently, uncertainty is represented by limits (intervals) on possible values of demand weights varying independently rather than by scenarios for all the weights simultaneously. In this paper we show that a solution concept of the conditional median can be used to effectively optimized such robust location problems. The conditional median is a generalization of the minimax solution concept extended to take into account the number of services (the portion of demand) related to the worst performances. Namely, for a specified portion of demand we take into account the corresponding portion of the maximum results and we consider their average as the worst conditional mean to be minimized. Similar to the standard minimax approach, the minimization of the worst conditional mean can be defined by a linear objective and a number of auxiliary linear inequalities.

Key Words. Location, Multiple Criteria, Efficiency, Robustness, Conditional Median.

1 Introduction

A conventional optimization approach that considers only the optimality of a decision or a design, that is, performance at decision or design condition, should work fine in a controlled environment. Real-world applications, on the other hand, inevitably involve errors and uncertainties in the operating conditions and thereby the resulting performance may be lower than expected. In particular, the distribution system design is very sensitive to the varying demands for goods and changed demands may deteriorate drastically the system efficiency when optimized for different demands structure.

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Several approaches have been developed to deal with uncertain or imprecise data. The approaches focused on the quality or on the variation (stability) of the solution for some data domains are considered robust. The notion of robustness applied to decision problems was first introduced almost 50 years ago by [3]. Practical importance of the performance sensitivity against data uncertainty and errors has later attracted considerable attention to the search for robust solutions. Actually, as suggested by [26], the concept of robustness should be applied not only to solutions but, more generally to various assertions and recommendations generated within a decision support process. A brief comparison between conventional optimization and robust optimization is illustrated in Fig. 1 [14]. Solution A obtained by a conventional optimization is the best in terms of optimality, but disperses widely in terms of the objective function against the dispersion of design variable or environmental variable, and this dispersion may extend to an infeasible range. However, solution B obtained by a robust optimization is moderately good in terms of optimality and also good in terms of robustness, that is, dispersion of objective function is narrow against dispersion of design variable. On the other hand, the optimal solution despite generating objective values dispersed quite widely may be clearly better than a solution not dispersed at all. As depicted in Fig. 1 b), Solution A though characterized by dispersed results remains under all conditions better than the stable Solution B. Hence, Solution B is obviously dominated and it cannot be considered a robust optimal solution.

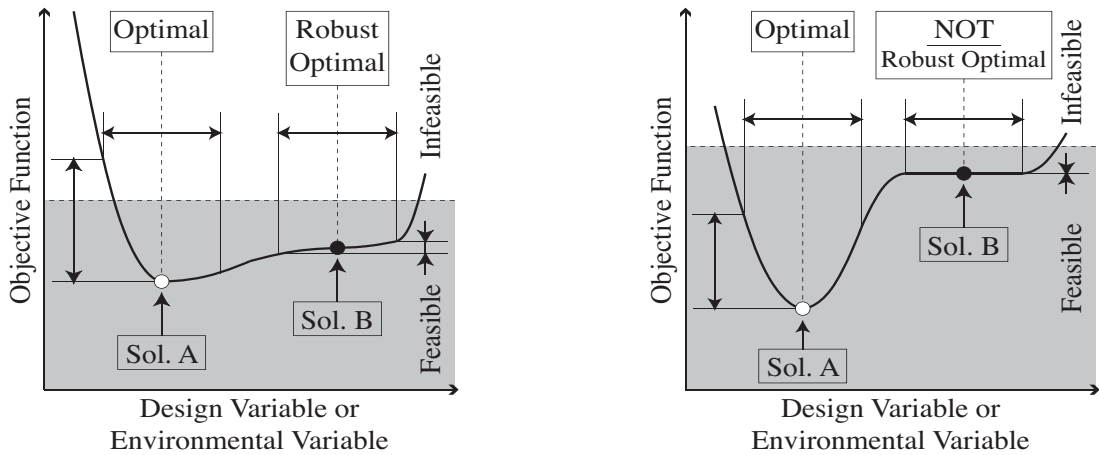


Figure 1: Conventional optimization and robust optimization (for a minimization problem): a) conventional optimal Solution A vs. robust optimal Solution B; b) stable but not robust optimal Solution B.

The precise concept of robustness depends on the way the uncertain data domains and the quality or stability characteristics are introduced. Typically, in robust analysis one does not attribute any probability distribution to represent uncertainties. Data uncertainty is rather represented by non-attributed scenarios. Since one wishes to optimize results under each scenario, robust optimization might be in some sense viewed as a multiobjective optimization problem where objectives correspond to the scenarios. However, despite of many similarities of such robust optimization concepts to multiobjective models, there are also some significant differences [4]. Actually, robust optimization is a problem of optimal distribution of objective values under several scenarios [18] rather than a standard multiobjective optimization model.

A conservative notion of robustness focusing on worst case scenario results is widely accepted and the min-max optimization is commonly used to seek robust solutions. The worst case scenario analysis can be applied either to the absolute values of objectives (the absolute robustness) or to the regret values (the deviational robustness) [8]. The latter, when considered from the multiobjective perspective, repre-

sents a simplified reference point approach with the utopian (ideal) objective values for all the scenario used as aspiration levels. Recently, a more advanced concept of ordered weighted averaging was introduced into robust optimization [24], thus, allowing to optimize combined performances under the worst case scenario together with the performances under the second worst scenario, the third worst and so on. Such an approach exploits better the entire distribution of objective vectors in search for robust solutions and, more importantly, it introduces some tools for modeling robust preferences. Actually, while more sophisticated concepts of robust optimization are considered within the area of discrete programming models, only the absolute robustness is usually applied to the majority of decision and design problems.

2 Location problem

The generic location problem that we consider may be stated as follows. There is given a set $I = \{1, 2, \dots, m\}$ of m clients (service recipients). Each client is represented by a specific point in the geographical space. There is also given a set Q of location patterns (location decisions). For each client i ($i \in I$) a function $f_i(\mathbf{x})$ of the location pattern $\mathbf{x}$ has been defined. This function, called the individual objective function, measures the outcome (effect) $y_i = f_i(\mathbf{x})$ of the location pattern for client i . [13]. In the simplest problems an outcome usually expresses the distance. However, we emphasize to the reader that we do not restrict our considerations to the case of outcomes measured as distances. They can be measured (modeled) as travel time, travel costs as well as in a more subjective way as relative travel costs (e.g., travel costs by clients incomes) or ultimately as the levels of clients dissatisfaction (individual disutility) of locations. In typical formulations of location problems related to desirable facilities a smaller value of the outcome (distance) means a better effect (higher service quality or client satisfaction). This remains valid for location of obnoxious facilities if the distances are replaced with their complements to some large number. Therefore, without loss of generality, we can assume that each individual outcome y_i is to be minimized.

Further, some additional client weights $w_i > 0$ are included into location model to represent the service demand. Integer weights can be interpreted as numbers of unweighted clients located at exactly the same place (with distances 0 among them). While discussing such solution concepts we will use the normalized client weights

$$\bar{w}_i = w_i / \sum_{i=1}^m w_i \quad \text{for } i = 1, 2, \dots, m \quad (1)$$

rather than the original quantities w_i . Note that, in the case of unweighted problem (all $w_i = 1$), all the normalized weights are given as $\bar{w}_i = 1/m$.

A host of operational models has been developed to deal with facility location optimization (c.f., [2, 10, 1]). Most classical location studies focus on the minimization of the mean (or total) distance (the median concept) or the minimization of the maximum distance (the center concept) to the service facilities [16]. Both the median and the center solution concepts are well defined for weighted location models using client weights $w_i > 0$ to represent several clients (service demand) at the same geographical point. Exactly, for the weighted location problem, the *median* solution concept is defined by minimization of the objective function expressing the *mean* (average) outcome

$$\mu(\mathbf{y}) = \sum_{i=1}^m \bar{w}_i y_i$$

but it is also equivalent to minimization of the total outcome $\sum_{i=1}^m w_i y_i$. The *center* solution concept is defined by minimization of the objective function representing the *maximum* (worst) outcome

$$M(\mathbf{y}) = \max_{i=1,\dots,m} y_i$$

and it is not affected by the client weights at all.

The median solution concept is primarily concerned with the spatial efficiency.

Assuming that the generic location problem that we consider has been disaggregated to the unweighted form (all $w_i = 1$), it may be stated as the following multiple criteria minimization problem:

$$\min \{ \mathbf{f}(\mathbf{x}) : \mathbf{x} \in Q \} \quad (2)$$

where $\mathbf{f} = (f_1, \dots, f_m)$ is a vector of the individual objective functions which measure the outcome (effect) $y_i = f_i(\mathbf{x})$ of the location pattern $\mathbf{x}$ for client i . A wide gamut of location problems can be considered within the framework of model (2). The following example illustrates how a typical discrete location problem [15] can be modeled in the form (2).

Example 1 In a typical discrete location problem there is given a set of m clients and a set of n potential locations for the facilities. Further, the number (or the maximal number) p of facilities to be located is given ($p \leq n$). The main decisions to be made can be described with the binary variables x_j ($j = 1, 2, \dots, n$) equal to 1 if location j is to be used and equal to 0 otherwise. To meet the problem requirements, the decision variables x_j have to satisfy the following constraints:

$$\sum_{j=1}^n x_j = p, \quad x_j \in \{0, 1\}, \quad \text{for } j = 1, 2, \dots, n. \quad (3)$$

For most location problems the feasible set has a more complex structure due to explicit consideration of allocation decisions. These decisions are usually modeled with the additional allocation variables x'_{ij} ($i = 1, 2, \dots, m; j = 1, 2, \dots, n$) equal to 1 if location j is used to service client i and equal to 0 otherwise. The allocation variables have to satisfy the following constraints:

$$\sum_{j=1}^n x'_{ij} = 1, \quad \text{for } i = 1, 2, \dots, m. \quad (4)$$

$$x'_{ij} \leq x_j, \quad \text{for } i = 1, 2, \dots, m \quad \text{and} \quad j = 1, 2, \dots, n. \quad (5)$$

$$x'_{ij} \in \{0, 1\}, \quad \text{for } i = 1, 2, \dots, m \quad \text{and} \quad j = 1, 2, \dots, n. \quad (6)$$

In the capacitated location problem the capacities of the potential facilities are given which implies some additional constraints.

Let $d_{ij} \geq 0$ ($i = 1, 2, \dots, m; j = 1, 2, \dots, n$) express the distance between client i and location j (or other effect of allocation client i to location j). With the explicit use of the allocation variables and the corresponding constraints (4)–(5) the individual objective functions f_i can be written in the following linear form:

$$f_i(\mathbf{x}) = \sum_{j=1}^n d_{ij} x'_{ij}, \quad \text{for } i = 1, 2, \dots, m. \quad (7)$$

These linear functions of the allocation variables are applicable for the uncapacitated as well as for the capacitated facility location problems.

In the case of location of desirable facilities a smaller value of the individual objective function means a better effect (smaller distance). This remains valid for location of obnoxious facilities if the distance coefficients are replaced with their complements to some large number: $d'_{ij} = d - d_{ij}$, where $d > d_{ij}$ for all $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$. Therefore, we can assume that each function f_i is to be minimized as stated in the multiple criteria problem (2). $\square$

We do not assume any special form of the feasible set while analyzing properties of the solution concepts. We rather allow the feasible set to be a general, possibly discrete (nonconvex), set. Similarly, we do not assume any special form of the individual objective functions nor their special properties (like convexity) while analyzing properties of the solution concepts. Therefore, the results of our analysis apply to various location problems.

The classical approaches to the location problems use either the median solution concept based on the minisum optimization:

$$\min \left\{ \sum_{i=1}^m \bar{w}_i f_i(\mathbf{x}) : \mathbf{x} \in Q \right\} \quad (8)$$

or the center solution concept defined by the minimax optimization:

$$\min \left\{ \max_{i=1, \dots, m} f_i(\mathbf{x}) : \mathbf{x} \in Q \right\}. \quad (9)$$

3 Robust locations

The median solution concept is usually very attractive due to maximizing the system efficiency. Let us define the set of attainable outcomes

$$A = \{ \mathbf{y} : y_i = f_i(\mathbf{x}) \forall i, \mathbf{x} \in Q \}$$

The median solution is then defined as

$$\min_{\mathbf{y} \in A} \left\{ \sum_{i=1}^m \bar{w}_i y_i \right\} \quad (10)$$

However, in practical problems the demand weights may vary. Therefore, a robust solution is sought which performs well under uncertain weights.

The simplest representation of uncertainty depends on a number of predefined scenarios $s = 1, \dots, r$. Let $\bar{w}_i^s$ denote the realization of demand i under scenario s . Then one may seek a robust solution by minimizing the mean distance under the worst scenario

$$\min_{\mathbf{y} \in A} \max_{s=1, \dots, r} \left\{ \sum_{i=1}^m \bar{w}_i^s y_i \right\} = \min_{\mathbf{y} \in A} \{ z : z \geq \sum_{i=1}^m \bar{w}_i^s y_i \forall s \} \quad (11)$$

or by minimizing the maximum regret

$$\min_{\mathbf{y} \in A} \max_{s=1, \dots, r} \left\{ \sum_{i=1}^m \bar{w}_i^s y_i - \bar{d}^s \right\} = \min_{\mathbf{y} \in A} \{ z : z \geq \sum_{i=1}^m \bar{w}_i^s y_i - \bar{d}^s \forall s \} \quad (12)$$

where $\bar{d}^s$ represent the best value under scenario s

$$\bar{d}^s = \min_{\mathbf{y} \in A} \left\{ \sum_{i=1}^m \bar{w}_i^s y_i \right\} \quad (13)$$

Frequently, uncertainty is represented by limits (intervals) on possible values of weights varying independently rather than by scenarios for all the weights simultaneously. Assume that the demand weights $\bar{w}_i$ may be affected by perturbations varying within intervals $[-\delta_i, \Delta_i]$. Note that the weights are normalized and although varying independently they must total to 1. Alternatively one may consider completely independent perturbations of unnormalized weights w_i and normalize them later. The worst case mean distance can be expressed then as

$$\mu^w(\mathbf{y}) = \max_{u_i} \left\{ \sum_{i=1}^m y_i u_i : \sum_{i=1}^m u_i = 1, \bar{w}_i - \delta_i \leq u_i \leq \bar{w}_i + \Delta_i \forall i \right\} \quad (14)$$

and the robust solution is defined by the minimization

$$\min_{\mathbf{y} \in A} \mu^w(\mathbf{y}) \quad (15)$$

Note that in the case of $\delta_i = \Delta_i = 0$ one gets the standard mean outcome $\mu^w(\mathbf{y}) = \sum_{i=1}^m y_i \bar{w}_i$. On the other hand, for the case of unlimited perturbations ($\delta_i = \bar{w}_i$ and $\Delta_i = 1 - \bar{w}_i$) The worst case mean outcome becomes the worst outcome $\mu^w(\mathbf{y}) = \max_{i=1, \dots, m} y_i$.

For the special case of proportional perturbation limits $\delta_i = \delta \bar{w}_i$ and $\Delta_i = \Delta \bar{w}_i$ with positive parameters δ and Δ , one gets

$$\begin{aligned} \mu^w(\mathbf{y}) &= \max_{u_i} \left\{ \sum_{i=1}^m y_i u_i : \sum_{i=1}^m u_i = 1, \bar{w}_i(1 - \delta) \leq u_i \leq \bar{w}_i(1 + \Delta) \forall i \right\} \\ &= (1 + \Delta) \max_{u'_i} \left\{ \sum_{i=1}^m y_i u'_i : \sum_{i=1}^m u'_i = 1/(1 + \Delta), \bar{w}_i(1 - \delta)/(1 + \Delta) \leq u'_i \leq \bar{w}_i \forall i \right\} \end{aligned} \quad (16)$$

In particular, when lower limits relaxed $\delta = 1$ this becomes the classical conditional mean outcome [22, 23]:

$$\mu_\beta(\mathbf{y}) = \frac{1}{\beta} \max_{u_i} \left\{ \sum_{i=1}^m y_i u_i : \sum_{i=1}^m u_i = \beta, 0 \leq u_i \leq \bar{w}_i \forall i \right\} \quad (17)$$

with $\beta = 1/(1 + \Delta)$ while the robust solution itself is the conditional median. Thus the conditional median represents a concept of a robust solution for proportionally upper bounded perturbations.

The conditional median is easily defined by LP constraints. Although similar simple form may be achieved for general worst case mean distance (14). Note that problem (14) is an LP for a given outcome vector $\mathbf{y}$ while it becomes nonlinear for $\mathbf{y}$ being a vector of variables. This difficulty can be overcome by taking advantage of the LP dual to (14). Introducing dual variable t corresponding to the equation $\sum_{i=1}^m u_i = 1$ and variables d_i^u and d_i^l corresponding to upper and lower bounds on u_i , respectively, one gets the following LP dual to problem (14):

$$\mu^w(\mathbf{y}) = \min_{t, d_i^u, d_i^l} \left\{ t + \sum_{i=1}^m (\bar{w}_i + \Delta_i) d_i^u - \sum_{i=1}^m (\bar{w}_i - \delta_i) d_i^l : t + d_i^u - d_i^l \geq y_i, d_i^u, d_i^l \geq 0 \forall i \right\} \quad (18)$$

and the entire robust solution problem

$$\min_{\mathbf{y} \in Y, t, d_i^u, d_i^l} \left\{ t + \sum_{i=1}^m (\bar{w}_i + \Delta_i) d_i^u - \sum_{i=1}^m (\bar{w}_i - \delta_i) d_i^l : t + d_i^u - d_i^l \geq y_i, d_i^u, d_i^l \geq 0 \forall i \right\} \quad (19)$$

In the case of relaxed lower limits ($\delta = 1$) this becomes the classical LP computational formula for the conditional mean outcome [22, 23]

$$\min_{\mathbf{y} \in Y, t, d_i^u} \left\{ \beta t + \sum_{i=1}^m \bar{w}_i d_i^u : t + d_i^u \geq y_i, d_i^u \geq 0 \quad \forall i \right\} \quad (20)$$

with $\beta = 1/(1 + \Delta)$.

Conclusions

- Mean outcome minimization (so-called Median solution concept in location problems) is well suited for system efficiency maximization.
- Serious uncertainty regarding demand weights may significantly lower efficiency of the selected design (location pattern). This causes a need for some robust solution concepts.
- Uncertainty represented by limits (intervals) on possible values of weights varying independently rather than by scenarios for all the weights simultaneously leads to the worst case mean optimization models with variable coefficients (weights).
- Unbounded weights perturbations lead us to the standard maxmin model (so-called center solution concept). The latter is usually too restrictive thus lowering the overall system efficiency.
- Proportional limits on weights perturbations lead us to minimization of the mean in β portion of the worst outcomes (so-called conditional median solution concept). The latter is usually less restrictive than the center solution while similarly implementable with auxiliary linear inequalities.
- Similar generalized conditional median model can be introduced for the case of arbitrary perturbations limits. Initial computational experiments on randomly generated discrete location problems confirm theoretical attractiveness of such robust optimization model.

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