

Report of the Institute of Control and Computation Engineering Warsaw University of Technology

Ordered Achievements Refinements of the Reference Point Method

Włodzimierz Ogryczak

October, 2007

Report nr: 07-06

Copyright ©2007 by the Institute of Control & Computation Engineering

Permission to use, copy and distribute this document for any purpose and without fee is hereby granted, provided that the above copyright notice and this permission notice appear in all copies, and that the name of ICCE not be used in advertising or publicity pertaining to this document without specific, written prior permission. ICCE makes no representations about the suitability of this document for any purpose. It is provided “as is” without express or implied warranty.



Warsaw University of Technology
Institute of Control & Computation Engineering
ul. Nowowiejska 15/19, 00-665 Warsaw, Poland
phone (48-22)-660-7397, fax (48-22)-825-3719



Ordered Achievements Refinements of the Reference Point Method*

Włodzimierz Ogryczak

ogryczak@ia.pw.edu.pl

October, 2007

Abstract

The Reference Point Method (RPM) is an interactive technique for an open search for a satisficing efficient solution to multi-criteria decision problems. The RPM is based on the so-called augmented max-min aggregation where the worst individual achievement maximization process is additionally regularized with the the average achievement. The regularization by the average achievement is easily implementable but it may disturb the basic max-min model. The OWA regularization applied to the RPM allows one to overcome this flaw since taking into account differences among all ordered achievement values. Further, allowing to introduce importance weights we get the WOWA enhanced RPM. The WOWA regularization is more complicated in implementation but the recent progress made in optimization methods for ordered averages allows one to implement the corresponding RPM method quite effectively.

Key Words. Multicriteria Decision Making, Aggregation Methods, Reference Point Method, Lex-min Order, OWA, WOWA.

1 Introduction

While dealing with a decision problem formulated in the multiple criteria optimization framework, it is clear that an outcome vector is better than another if all of its individual outcome are better or at least one individual outcome is better whereas no other one is worse. Unfortunately, there usually does not exist an outcome vector that dominates in such a sense all the others. Thus in terms of strict mathematical relations one cannot distinguish the best outcome vector. The nondominated vectors are incomparable on the basis of the specified set of criteria. The decisions that generate nondominated outcome vectors are called efficient or Pareto-optimal solutions to the multiple criteria problem. For an operational use of multiple criteria analysis, it must help the Decision Maker (DM) to select one efficient solution for implementation. Certainly, the original criteria do not allow one to select any efficient solution as better than any other one. Therefore, the decision support process must depend on additional preference information gained from the DM.

*This work was partially supported by The Ministry of Science and Information Society Technologies under the grant KBN-3T11C00527 "Models and Algorithms for Efficient and Fair Resource Allocation in Complex Systems."

Typical multiple criteria optimization methods aggregate the individual outcomes with some scalarizing functions to generate a satisfactory efficient solution. The scalarizing functions may have various constructions and properties depending on the specific approach to preference modeling applied in several methods. Nevertheless, most scalarizing functions can be viewed as two-stage transformation of the original outcomes. First the individual outcomes are rescaled to some uniform measures of achievements with respect to several criteria and preference parameters. Thus, the individual achievement functions are built to measure actual achievement of each outcome with respect to the corresponding preference parameters. In particular, in the Reference Point Method (RPM) the strictly monotonic partial achievement functions are built to measure individual performance with respect to given reference levels. Similar constructions appear in fuzzy approaches where the membership functions for various fuzzy targets are such individual achievement measures scaled to the unit interval or in goal programming where scaled deviations from targets may be considered individual achievements.

Having all the outcomes transformed into a uniform scale of individual achievements they are aggregated at the second stage to form a unique scalarization. The aggregation usually measures the total (the average) or the worst individual achievement. While several techniques and tools for better modeling of preferences with partial achievement functions are developed [6], the aggregation itself is much less studied. The RPM is based on the so-called augmented (or regularized) max-min aggregation. Thus, the worst individual achievement is essentially maximized but the optimization process is additionally regularized with the term representing the average achievement. The max-min aggregation guarantees fair treatment of all individual achievements by implementing an approximation to the Rawlsian principle of justice. The max-min aggregation is crucial for allowing the RPM to generate all efficient solutions even for nonconvex (and particularly discrete) problems. On the other hand, the regularization is necessary to guarantee that only efficient solution are generated. The regularization by the average achievement is easily implementable but it may disturb the basic max-min model. Actually, the only consequent regularization of the max-min aggregation is the lexicographic max-min (nucleolar) solution concept where in addition to the worst achievement, the second worst achievement is also optimized (provided that the worst remains on the optimal level), the third worst is optimized (provided that the two worst remain optimal), and so on. Such a nucleolar regularization is the only max-min regularization satisfying the addition/deleting principle thus making the corresponding nucleolar RPM not affected by any passive criteria.

The OWA aggregation with monotonic weights combines all the partial achievements allocating the largest weight to the worst achievement, the second largest weight to the second worst achievement, the third largest weight to the third worst achievement, and so on. The recent progress in optimization methods for ordered averages [22] allows one to implement the OWA RPM quite effectively. Further following the concept of Weighted OWA [27] the importance weighting of several achievements may be incorporated into the RPM. Such a WOWA enhancement of the RPM uses importance weights to affect achievement importance by rescaling accordingly its measure within the distribution of achievements rather than straightforward rescaling of achievement values [26]. The paper analyzes both the theoretical and implementation issues of the WOWA enhanced RPM.

2 Scalarizations of the Reference Point Method

In this paper, without loss of generality, it is assumed that all the criteria are maximized (that is, for each outcome ‘more is better’). Hence, we consider the following multiple criteria optimization problem:

$$\max \{ (f_1(\mathbf{x}), f_2(\mathbf{x}), \dots, f_m(\mathbf{x})) : \mathbf{x} \in Q \} \quad (1)$$

where $\mathbf{x}$ denotes a vector of decision variables to be selected within the feasible set $Q \subset R^n$, and $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \dots, f_m(\mathbf{x}))$ is a vector function that maps the feasible set Q into the criterion space R^m . Note that neither any specific form of the feasible set Q is assumed nor any special form of criteria $f_i(\mathbf{x})$ is required. We refer to the elements of the criterion space as outcome vectors. An outcome vector $\mathbf{y}$ is attainable if it expresses outcomes of a feasible solution, i.e. $\mathbf{y} = \mathbf{f}(\mathbf{x})$ for some $\mathbf{x} \in Q$.

Model (1) only specifies that we are interested in maximization of all objective functions f_i for $i \in I = \{1, 2, \dots, m\}$. Thus it allows only to identify (to eliminate) obviously inefficient solutions leading to dominated outcome vectors, while still leaving the entire efficient set to look for a satisfactory compromise solution. In order to make the multiple criteria model operational for the decision support process, one needs assume some solution concept well adjusted to the DM preferences. This can be achieved with the so-called quasi-satisficing approach to multiple criteria decision problems. The best formalization of the quasi-satisficing approach to multiple criteria optimization was proposed and developed mainly by Wierzbicki [30] as the reference point method. The reference point method was later extended to permit additional information from the DM and, eventually, led to efficient implementations of the so-called aspiration/reservation based decision support (ARBDS) approach with many successful applications [7, 31].

The RPM is an interactive technique. The basic concept of the interactive scheme is as follows. The DM specifies requirements in terms of reference levels, i.e., by introducing reference (target) values for several individual outcomes. Depending on the specified reference levels, a special scalarizing achievement function is built which may be directly interpreted as expressing utility to be maximized. Maximization of the scalarizing achievement function generates an efficient solution to the multiple criteria problem. The computed efficient solution is presented to the DM as the current solution in a form that allows comparison with the previous ones and modification of the reference levels if necessary.

While building the scalarizing achievement function the following properties of the preference model are assumed. First of all, for any individual outcome y_i more is preferred to less (maximization). To meet this requirement the function must be strictly increasing with respect to each outcome. Second, a solution with all individual outcomes y_i satisfying the corresponding reference levels is preferred to any solution with at least one individual outcome worse (smaller) than its reference level. That means, the scalarizing achievement function maximization must enforce reaching the reference levels prior to further improving of criteria. Thus, similar to the goal programming approaches, the reference levels are treated as the targets but following the quasi-satisficing approach they are interpreted consistently with basic concepts of efficiency in the sense that the optimization is continued even when the target point has been reached already.

The generic scalarizing achievement function takes the following form [30]:

$$S(\mathbf{y}) = \min_{1 \leq i \leq m} \{s_i(y_i)\} + \frac{\varepsilon}{m} \sum_{i=1}^m s_i(y_i) \quad (2)$$

where ε is an arbitrary small positive number and $s_i : R \rightarrow R$, for $i = 1, 2, \dots, m$, are the partial achievement functions measuring actual achievement of the individual outcomes y_i with respect to the corresponding reference levels. Let a_i denote the partial achievement for the i th outcome ($a_i = s_i(y_i)$) and $\mathbf{a} = (a_1, a_2, \dots, a_m)$ represent the achievement vector. The scalarizing achievement function (2) is, essentially, defined by the worst partial (individual) achievement but additionally regularized with the sum of all partial achievements. The regularization term is introduced only to guarantee the solution efficiency in the case when the maximization of the main term (the worst partial achievement) results in a non-unique optimal solution. Due to combining two terms with arbitrarily small parameter ε , formula (2)

is easily implementable and it provides a direct interpretation of the scalarizing achievement function as expressing utility. When accepting the loss of a direct utility interpretation, one may consider a limiting case with $\varepsilon \rightarrow 0_+$ which results in lexicographic order applied to two separate terms of function (2). That means, the regularization can be implemented with the second level lexicographic optimization [18]. Therefore, RPM may be also considered as the following lexicographic problem ([16] and references therein):

$$\text{lex max } \{ (\min_{1 \leq i \leq m} a_i, \sum_{i=1}^m a_i) : a_i = s_i(f_i(\mathbf{x})) \forall i, \mathbf{x} \in Q \} \quad (3)$$

The following two properties of the lexicographic model (3) are crucial for the RPM methodology:

P1: The aggregation is strictly monotonic in the sense that increase of any partial achievement a_i leads to a preferred solution.

P2: For any given target value ϱ , the solution generating all partial achievements equal to ϱ ($a_i = \varrho \forall i$) is preferred to any solution generating at least one partial achievement worse than ϱ .

Property P1 guarantees that while using strictly increasing partial achievement functions s_i , every generated solution is efficient. Property P2 guarantees that while using partial achievement function allocating the same value on achieving the reference level, the solution reaching all the reference levels is preferred to any solution failing achievement of at least one reference level.

Various functions s_i provide a wide modeling environment for measuring partial achievements [31, 12]. To take advantages of properties P1 and P2 they need to be strictly increasing and to allocate the same value on reaching the reference level. The basic RPM model is based on a single vector of the reference levels, the aspiration vector $\mathbf{r}^a$. For the sake of computational simplicity, the piecewise linear functions s_i are usually employed. In the simplest models, they take a form of two segment piecewise linear functions:

$$s_i(y_i) = \begin{cases} \lambda_i^+(y_i - r_i^a), & \text{for } y_i \geq r_i^a \\ \lambda_i^-(y_i - r_i^a), & \text{for } y_i < r_i^a \end{cases} \quad (4)$$

where λ_i^+ and λ_i^- are positive scaling factors corresponding to underachievements and overachievements, respectively, for the i th outcome. Note that for any outcome reaching the corresponding aspiration level $y_i = r_i^a$ one gets $s_i(r_i^a) = 0$. Hence, when using the RPM (3) with partial achievement functions (4), the solution reaching all the aspiration levels is preferred to any solution failing achievement of at least one aspiration level. It is usually assumed that λ_i^+ is much larger than λ_i^- . Actually, even linear functions

$$s_i(y_i) = \lambda_i(y_i - r_i^a) \quad (5)$$

with positive scaling factors λ_i represent simplified (but still valid) partial achievement functions in the sense that while used in the lexicographic RPM scheme (3) it guarantees the property **P2**. Nevertheless, the differentiation of the scaling factor is important to enforce the preferences of achieving more aspiration levels rather than overstep the others, especially in the analytic RPM (2). Figure 1 depicts how differentiated scaling affects the isoline contours of the analytic scalarizing achievement function. Certainly, introducing lexicographic two-level partial achievements optimization would be a better way to model the aspiration properties [14] but also more complicated.

Real-life applications of the RPM methodology usually deal with more complex partial achievement functions defined with more than one reference point [31] which enriches the preference models and simplifies the interactive analysis. In particular, the models taking advantages of two reference vectors: vector of aspiration levels $\mathbf{r}^a$ and vector of reservation levels $\mathbf{r}^r$ [7] are used, thus allowing the DM to specify requirements by introducing acceptable and required values for several outcomes. The partial

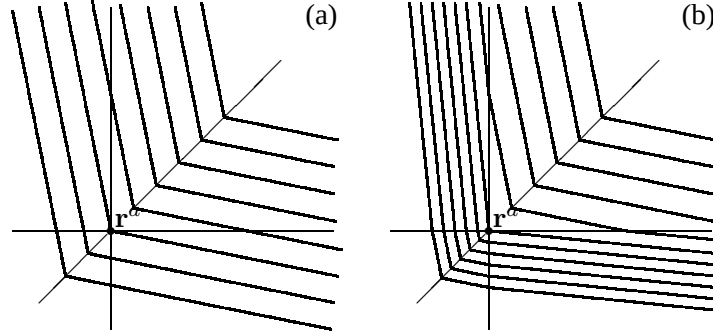


Figure 1: Isoline contours for the analytic scalarizing achievement function (2): (a) with partial achievements (5), (b) with partial achievements (4)

achievement function s_i can be interpreted then as a measure of the DM's satisfaction with the current value of outcome the i th criterion. It is a strictly increasing function of outcome y_i with value $a_i = 1$ if $y_i = r_i^a$, and $a_i = 0$ for $y_i = r_i^r$. Thus the partial achievement functions map the outcomes values onto a normalized scale of the DM's satisfaction. Various functions can be built meeting those requirements. We use the piece-wise linear partial achievement function introduced in an implementation of the ARBDS system for the multiple criteria transshipment problems with facility location [20]:

$$s_i(y_i) = \begin{cases} \gamma & (y_i - r_i^r)/(r_i^a - r_i^r), & \text{for } y_i \leq r_i^r \\ (y_i - r_i^r)/(r_i^a - r_i^r), & \text{for } r_i^r < y_i < r_i^a \\ \alpha & (y_i - r_i^a)/(r_i^a - r_i^r) + 1, & \text{for } y_i \geq r_i^a \end{cases} \quad (6)$$

where α and γ are arbitrarily defined parameters satisfying $0 < \alpha < 1 < \gamma$. Parameter α represents additional increase of the DM's satisfaction over level 1 when a criterion generates outcomes better than the corresponding aspiration level. On the other hand, parameter $\gamma > 1$ represents dissatisfaction connected with outcomes worse than the reservation level.

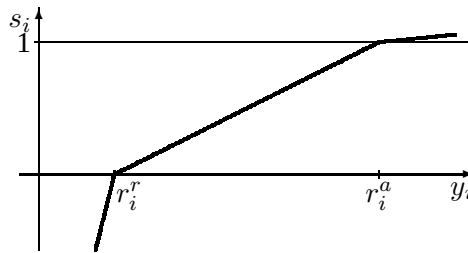


Figure 2: ARBDS partial achievement function

For outcomes between the reservation and the aspiration levels, the partial achievement function s_i can be interpreted as a membership function μ_i for a fuzzy target. However, such a membership function remains constant with value 1 for all outcomes greater than the corresponding aspiration level, and with value 0 for all outcomes below the reservation level (Fig. 2). Hence, the fuzzy membership function is neither strictly monotonic nor concave thus not representing typical utility for a maximized outcome. The partial achievement function (6) can be viewed as an extension of the fuzzy membership function to a strictly monotonic and concave utility. One may also notice that the aggregation scheme used to build the scalarizing achievement function (2) from the partial ones may also be interpreted as some fuzzy

aggregation operator [31]. In other words, maximization of the scalarizing achievement function (2) is consistent with the fuzzy methodology in the case of not attainable aspiration levels and satisfiable all reservation levels while modeling a reasonable utility for any values of aspiration and reservation levels.

3 Nucleolar RPM

The crucial properties of the RPM are related to the max-min aggregation of partial achievements while the regularization is only introduced to guarantee the aggregation monotonicity. Unfortunately, the distribution of achievements may make the max-min criterion partially passive when one specific achievement is relatively very small for all the solutions. Maximization of the worst achievement may then leave all other achievements unoptimized. In the lexicographic RPM defined by (3) the regularization term is then optimized on the second level thus preventing one from selection of any inefficient solution. Nevertheless, the selection is then made according to linear aggregation of the regularization term instead of the max-min aggregation, thus destroying the preference model of the RPM. This can be illustrated with an example of a simple discrete problem of 7 alternative feasible solutions to be selected according to 6 criteria. Table 1 presents six partial achievements for all the solutions where the partial achievements have been defined according to the aspiration/reservation model (6) thus allocating 1 to outcomes reaching the corresponding aspiration level. Solution S7 is the only inefficient alternative. Solution S1 to S5 oversteps the aspiration levels (achievement values 1.2) for four of the first five criteria while failing to reach one of them and the aspiration level for the sixth criterion as well (achievement values 0.3). Solution S6 meets the aspiration levels (achievement values 1.0) for the first five criteria while failing to reach only the aspiration level for the sixth criterion (achievement values 0.3). One may easily notice that the sixth partial achievement (and the corresponding criterion) is constant for the seven alternatives under consideration. Hence, one may expect the same solution selected while taking into account this criterion or not. If focusing on only five first criteria, then the RPM (either lexicographic (3) or analytic (2)) obviously selects solution S6 as reaching all aspiration levels which results in the worst achievement value 1.0. However, while taking into account all six criteria all the solutions generate the same worst achievement value 0.3 and the final selection of the RPM depends on the total achievement (regularization term). Actually, either lexicographic RPM (3) or its analytic version (2) will select then one of solutions S1 to S5 as better than S6.

Table 1: Sample achievements with a passive criterion

Solution	a_1	a_2	a_3	a_4	a_5	a_6	$\min_{i=1,\dots,6}$	$\sum_{i=1}^6$	$\min_{i=1,\dots,5}$	$\sum_{i=1}^5$
S1	0.3	1.2	1.2	1.2	1.2	0.3	0.3	5.4	0.3	5.1
S2	1.2	0.3	1.2	1.2	1.2	0.3	0.3	5.4	0.3	5.1
S3	1.2	1.2	0.3	1.2	1.2	0.3	0.3	5.4	0.3	5.1
S4	1.2	1.2	1.2	0.3	1.2	0.3	0.3	5.4	0.3	5.1
S5	1.2	1.2	1.2	1.2	0.3	0.3	0.3	5.4	0.3	5.1
S6	1.0	1.0	1.0	1.0	1.0	0.3	0.3	5.3	1.0	5.0
S7	0.3	1.0	0.3	1.0	0.6	0.3	0.3	3.5	0.3	3.2

In order to avoid inconsistencies caused by the regularization, the max-min solution may be regularized according to the Rawlsian principle of justice. Formalization of this concept leads us to the lexicographic max-min ordered or nucleolar solution concept. The approach has been used for general linear programming multiple criteria problems [1, 11] as well as for specialized problems related

to (multi)period) resource allocation [9]. In discrete optimization it has been considered for various problems including the location-allocation ones [13]. The lexicographic max-min approach can be mathematically formalized as follows. Within the space of achievement vectors we introduce map $\Theta = (\theta_1, \theta_2, \dots, \theta_m)$ which orders the coordinates of achievements vectors in a nondecreasing order, i.e., $\Theta(a_1, a_2, \dots, a_m) = (\theta_1(\mathbf{a}), \theta_2(\mathbf{a}), \dots, \theta_m(\mathbf{a}))$ iff there exists a permutation τ such that $\theta_i(\mathbf{a}) = a_{\tau(i)}$ for all i and $\theta_1(\mathbf{a}) \leq \theta_2(\mathbf{a}) \leq \dots \leq \theta_m(\mathbf{a})$. The standard max-min aggregation depends on maximization of $\theta_1(\mathbf{a})$ and it ignores values of $\bar{a}_i$ for $i \geq 2$. In order to take into account all the achievement values, we look for a lexicographic maximum among the ordered achievement vectors.

Note that the lexicographic RPM model (3) can be expressed as the following problem:

$$\text{lex max } \{ (\theta_1(\mathbf{a}), \sum_{i=2}^m \theta_i(\mathbf{a})) : a_i = s_i(f_i(\mathbf{x})) \forall i, \mathbf{x} \in Q \}$$

thus, in the case of two criteria ($m = 2$), representing exactly the lexicographic max-min aggregation. For larger number of criteria ($m > 2$) model (3) only approximates the lexicographic max-min as all the lower priority objective terms are aggregated at the second priority level. One may consider the lexicographic max-min approach applied to the partial achievement functions (7) as a basis for a corresponding nucleolar RPM model

$$\text{lex max } \{ (\theta_1(\mathbf{a}), \theta_2(\mathbf{a}), \dots, \theta_m(\mathbf{a})) : a_i = s_i(f_i(\mathbf{x})) \forall i, \mathbf{x} \in Q \} \quad (7)$$

We will use the name nucleolar RPM to avoid any possible misunderstandings when referring to the lexicographic RPM. The nucleolar RPM implements a consequent max-min aggregation thus resulting in much better modeling of the reference levels concept.

Table 2: Ordered achievements values

Solution	$\theta_1(\mathbf{a})$	$\theta_2(\mathbf{a})$	$\theta_3(\mathbf{a})$	$\theta_4(\mathbf{a})$	$\theta_5(\mathbf{a})$	$\theta_6(\mathbf{a})$
S1	0.3	0.3	1.2	1.2	1.2	1.2
S2	0.3	0.3	1.2	1.2	1.2	1.2
S3	0.3	0.3	1.2	1.2	1.2	1.2
S4	0.3	0.3	1.2	1.2	1.2	1.2
S5	0.3	0.3	1.2	1.2	1.2	1.2
S6	0.3	1.0	1.0	1.0	1.0	1.0
S7	0.3	0.3	0.3	0.6	1.0	1.0

One may easily notice that the nucleolar RPM is not affected by any adding or eliminating any passive criterion. While applying the nucleolar RPM the ordered achievement are lexicographically minimized and therefore in our example solution S6 is selected for six criteria as it was selected for five criteria (Table 2). Actually, the lexicographic max-min is the only regularization of the max-min approach satisfying the reduction (addition/deleting) principle [3]. Namely, if the individual achievement of an outcome does not distinguish two solutions, then it does not affect the preference relation:

$$(a'_1, \dots, a'_i, a^*, a'_{i+1}, \dots, a'_q) \succeq (a''_1, \dots, a''_i, a^*, a''_{i+1}, \dots, a''_q) \Leftrightarrow a' \succeq a'' \quad (8)$$

Due to strictly monotonic individual achievement functions, the reduction principle is also satisfied in the original outcome space. Moreover, since the aggregation is impartial with respect to partial achievements, it depends only on distribution of achievements independently from their order. Hence, the nucleolar RPM works also properly if the max-min optimization becomes passive despite one cannot identify

any passive original criterion. This can be illustrated with data from Table 3 which differ from those of Table 1 only due to permuted achievements of solution S7. This alternative is no longer dominated and the sixth criterion is no longer passive. Nevertheless, as the distributions of achievement values remain the same, the max-min optimization remains passive and the standard forms of the RPM select solution S1 to S5 according to regularization term. Similarly, the ordered values of achievements remain the same as in Table 2, and the nucleolar RPM still selects solution S6 as the best matching the aspiration levels.

Table 3: Sample achievements with passive max-min criterion

Solution	a_1	a_2	a_3	a_4	a_5	a_6
S1	0.3	1.2	1.2	1.2	1.2	0.3
S2	1.2	0.3	1.2	1.2	1.2	0.3
S3	1.2	1.2	0.3	1.2	1.2	0.3
S4	1.2	1.2	1.2	0.3	1.2	0.3
S5	1.2	1.2	1.2	1.2	0.3	0.3
S6	1.0	1.0	1.0	1.0	1.0	0.3
S7	0.3	0.3	0.3	1.0	0.6	1.0

The following assertions shows that the nucleolar RPM model (7) satisfies the basic requirements for the RPM approaches. Namely, model (7) guarantees the efficiency of solutions (Theorem 1) and it is possible to generate all efficient solutions using nucleolar RPM by appropriately choosing the reference vector (Theorem 2).

Theorem 1 *For any strictly increasing partial achievement functions $s_i(y_i)$, if $\bar{x}$ is an optimal solution of the problem (7), then $\bar{x}$ is also an efficient solution of the corresponding multi-criteria problem (1).*

Proof. Suppose $\bar{x}$ optimal to (7) is dominated by some $x' \in Q$. Thus, due to strictly increasing partial achievement functions one gets $\bar{a}_i = s_i(f_i(\bar{x})) \leq a'_i = s_i(f_i(x')) \forall i$, with at least one inequality strict. Hence, $\Theta(\bar{a}) <_{lex} \Theta(a')$ which contradicts optimality $\bar{x}$ to (7). $\square$

Theorem 2 *For any $\bar{x}$ efficient solution, if the reference level are defined as $r_i = f_i(\bar{x})$ and strictly increasing partial achievement functions s_i taking the same value at the reference levels ($s_i(r_i) = \varrho \forall i$) are used, then $\bar{x}$ is an optimal solution of the corresponding nucleolar RPM problem (7).*

Proof. Note that $\bar{a}_i = s_i(f_i(\bar{x})) = \varrho \forall i$. If there exist $a'_i = s_i(f_i(x'))$ for $x' \in Q$ such that $\Theta(\bar{a}) <_{lex} \Theta(a')$, then $\bar{a}_i \leq a'_i \forall i$ with at least one inequality strict. This contradicts the efficiency of $\bar{x}$. $\square$

Note that all typical partial achievement functions, in particular piecewise linear functions of the form (4), (5) or (6) are strictly increasing and they assign the same value at the reference levels. Thus, Theorem 2 justifies the controllability of the nucleolar RPM in the sense that for any $x \in Q$ efficient solution to multiple criteria problem (1) there exists the reference vector r^a such that x is an optimal solution of the corresponding nucleolar RPM problem (7) defined with this reference vector.

4 OWA Refined Reference Point Method

An important advantage of the RPM depends on its easy implementation as an expansion of the original multiple criteria model. Actually, even more complicated partial achievement functions of the form (6)

are strictly increasing and concave (under the assumption that $0 < \alpha < 1 < \gamma$), thus allowing for implementation of the entire RPM model (2) by an LP expansion [20]. The ordered achievements optimized in the nucleolar RPM (7) are, in general, hard to implement due to the pointwise ordering. Let us consider cumulated achievements $\bar{\theta}_k(\mathbf{a}) = \sum_{i=1}^k \theta_i(\mathbf{a})$ expressing, respectively: the worst (smallest) achievement, the total of the two worst achievements, the total of the three worst achievements, etc. Within the lexicographic optimization a cumulation of criteria does not affect the optimal solution. Hence, the nucleolar RPM model (7) can be expressed in terms of the lexicographic maximization of quantities $\bar{\theta}_i(\mathbf{a})$:

$$\text{lex max } \{ (\bar{\theta}_1(\mathbf{a}), \bar{\theta}_2(\mathbf{a}), \dots, \bar{\theta}_m(\mathbf{a})) : a_i = s_i(f_i(\mathbf{x})) \ \forall i, \mathbf{x} \in Q \} \quad (9)$$

This simplifies dramatically the optimization problem since quantities $\bar{\theta}_k(\mathbf{a})$ can be optimized without use of any integer variables. First, let us notice that for any given vector $\mathbf{a}$, the cumulated ordered value $\bar{\theta}_k(\mathbf{a})$ can be found as the optimal value of the following LP problem:

$$\bar{\theta}_k(\mathbf{a}) = \min_{u_{ik}} \left\{ \sum_{i=1}^m a_i u_{ik} : \sum_{i=1}^m u_{ik} = k, \ 0 \leq u_{ik} \leq 1 \ \forall i \right\} \quad (10)$$

The above problem is an LP for a given outcome vector $\mathbf{a}$ while it becomes nonlinear for $\mathbf{a}$ being a vector of variables. This difficulty can be overcome by taking advantage of the LP dual to (10). Introducing dual variable t_k corresponding to the equation $\sum_{i=1}^m u_{ik} = k$ and variables d_{ik} corresponding to upper bounds on u_{ik} one gets the following LP dual of problem (10):

$$\bar{\theta}_k(\mathbf{a}) = \max_{t_k, d_{ik}} \left\{ kt_k - \sum_{i=1}^m d_{ik} : a_i \geq t_k - d_{ik}, \ d_{ik} \geq 0 \ \forall i \right\} \quad (11)$$

Due the duality theory, for any given vector $\mathbf{a}$ the cumulated ordered coefficient $\bar{\theta}_k(\mathbf{a})$ can be found as the optimal value of the above LP problem. It follows from (11) that $\bar{\theta}_k(\mathbf{a}) = \max \{ kt_k - \sum_{i=1}^m (t_k - a_i)_+ \}$ where $(\cdot)_+$ denotes the nonnegative part of a number and t_k is an auxiliary (unbounded) variable. The latter, with the necessary adaptation to the minimized outcomes in location problems, is equivalent to the computational formulation of the k -centrum model introduced by [24]. Hence, formula (11) provides an alternative proof of that formulation.

Taking advantages of both (9) and (11), the nucleolar RPM can be formulated as a standard lexicographic optimization. Moreover, in the case of concave piecewise linear partial achievement functions (as typically used in the RPM approaches), the resulting formulation extends the original constraints and criteria with linear inequalities. In particular, for strictly increasing and concave partial achievement functions (6), it can be expressed in the form:

$$\begin{aligned} & \text{lex max} && (z_1, z_2, \dots, z_m) \\ & \text{s.t.} && \mathbf{x} \in Q, \ y_i = f_i(\mathbf{x}) && \forall i \\ & && z_k = kt_k - \sum_{i=1}^m d_{ik} && \forall k \\ & && a_i \geq t_k - d_{ik}, \ d_{ik} \geq 0 && \forall i, k \\ & && a_i \leq \gamma(y_i - r_i^r)/(r_i^a - r_i^r) && \forall i \\ & && a_i \leq (y_i - r_i^r)/(r_i^a - r_i^r) && \forall i \\ & && a_i \leq \alpha(y_i - r_i^a)/(r_i^a - r_i^r) + 1 && \forall i \end{aligned} \quad (12)$$

Thus, the nucleolar RPM can be effectively applied to various multiple criteria optimization problems including the discrete ones.

Model (12) provides us with an easily implementable sequential algorithm to generate efficient solutions according to the nucleolar RPM preference specification. However, it does not introduce any explicit scalarizing achievement function which could be directly interpreted as expressing utility to be maximized. In order to get such an analytical form (or rather approximation) of the nucleolar RPM one needs to replace the lexicographic (preemptive) optimization of the ordered achievements in (7) with its weighting approximation. Note that the weights are then assigned to the specific positions within the ordered achievements rather than to the partial achievements themselves, thus representing the so-called Ordered Weighted Averaging (OWA) aggregation. Let $\mathbf{w} = (w_1, \dots, w_m)$ be a weighting vector of dimension m such that $w_i \geq 0$ for $i = 1, 2, \dots, m$ and $\sum_{i=1}^m w_i = 1$. The corresponding OWA aggregation of vector $\mathbf{a} = (a_1, \dots, a_m)$ can be mathematically formalized as follows [32]. First, we order non-increasingly the vector coordinates from the largest to the smallest one: $\langle \mathbf{a} \rangle = (a_{\langle 1 \rangle}, a_{\langle 2 \rangle}, \dots, a_{\langle m \rangle})$, where $a_{\langle 1 \rangle} \geq a_{\langle 2 \rangle} \geq \dots \geq a_{\langle m \rangle}$. Further, we apply the weighted sum aggregation to ordered achievement vectors $\langle \mathbf{a} \rangle$, i.e. the OWA aggregation has the following form:

$$A_{\mathbf{w}}(\mathbf{a}) = \sum_{i=1}^m w_i a_{\langle i \rangle} \quad (13)$$

The OWA aggregation (13) allows to model various aggregation functions from the maximum ($w_1 = 1$, $w_i = 0$ for $i = 2, \dots, m$) through the arithmetic mean ($w_i = 1/m$ for $i = 1, \dots, m$) to the minimum ($w_m = 1$, $w_i = 0$ for $i = 1, \dots, m-1$). The OWA aggregation may model various preferences from the optimistic (max) to the pessimistic (min). The OWA definition traditionally assume ordering from the largest to the smallest quantity. Therefore, $a_{\langle i \rangle} = \theta_{m-i+1}(\mathbf{a})$ and the OWA value may be equivalently expressed as

$$A_{\mathbf{w}}(\mathbf{a}) = \sum_{i=1}^m w_i \theta_{m-i+1}(\mathbf{a}) = \sum_{i=1}^m w_{m-i+1} \theta_i(\mathbf{a}) \quad (14)$$

With the OWA aggregation one gets the following RPM model:

$$\max \left\{ \sum_{i=1}^m v_i \theta_i(\mathbf{a}) : a_i = s_i(f_i(\mathbf{x})) \forall i, \mathbf{x} \in Q \right\} \quad (15)$$

where $v_1 > v_2 > \dots > v_m > 0$ are positive and strictly decreasing weights ($v_i = w_{m-i+1}$). When differences among weights tend to infinity, the OWA aggregation approximates the leximin ranking of the ordered outcome vectors [35]. That means, as the limiting case of (15), we get the nucleolar RPM model (7). Actually, the standard RPM model with the analytic scalarizing achievement function (2) can be expressed as the following OWA model:

$$\max \left\{ \left(1 + \frac{\varepsilon}{m}\right) \theta_1(\mathbf{a}) + \frac{\varepsilon}{m} \sum_{i=2}^m \theta_i(\mathbf{a}) : a_i = s_i(f_i(\mathbf{x})) \forall i, \mathbf{x} \in Q \right\}$$

Hence, the standard RPM model exactly represents the analytic (utility) form of the OWA aggregation (15) with strictly decreasing weights in the case of $m = 2$ ($v_1 = 1 + \varepsilon/2 > v_2 = \varepsilon/2$). For $m > 2$ it abandons the differences in weighting of the second largest achievement, the third largest one etc ($v_2 = \dots = v_m = \varepsilon/m$). The OWA RPM model 15 allows one to distinguish all the weights by introducing decreasing series (e.g. geometric ones). One may notice in Table 4 that application of decreasing weights $\mathbf{w} = (0.5, 0.25, 0.15, 0.05, 0.03, 0.02)$ within the OWA RPM enable selection of solution S6 from Table3.

Table 4: Ordered achievements values

Sol.	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	A_w
S1	0.3	0.3	1.2	1.2	1.2	1.2	0.525
S2	0.3	0.3	1.2	1.2	1.2	1.2	0.525
S3	0.3	0.3	1.2	1.2	1.2	1.2	0.525
S4	0.3	0.3	1.2	1.2	1.2	1.2	0.525
S5	0.3	0.3	1.2	1.2	1.2	1.2	0.525
S6	0.3	1.0	1.0	1.0	1.0	1.0	0.650
S7	0.3	0.3	0.3	0.6	1.0	1.0	0.305
w	0.5	0.25	0.15	0.05	0.03	0.02	

The OWA aggregation is obviously a piecewise linear function since it remains linear within every area of the fixed order of arguments. Its optimization can be implemented by expressing in terms of the cumulated ordered achievements

$$\max \left\{ \sum_{i=1}^m w_i \bar{\theta}_i(\mathbf{a}) : a_i = s_i(f_i(\mathbf{x})) \forall i; \mathbf{x} \in Q \right\}$$

where $w'_i = v_i - v_{i+1}$ for $i = 1, \dots, m-1$, and taking advantages of the LP expression (11) of $\bar{\theta}_i$ [22]. This leads to a single level computational model similar to (12).

$$\begin{aligned} \max \sum_{k=1}^m w'_k z_k \quad \text{s.t.} \quad & z_k = kt_k - \sum_{i=1}^m d_{ik} & \forall k \\ & \mathbf{x} \in Q, y_i = f_i(\mathbf{x}) & \forall i \\ & a_i \geq t_k - d_{ik}, d_{ik} \geq 0 & \forall i, k \\ & a_i \leq \gamma(y_i - r_i^r)/(r_i^a - r_i^r) & \forall i \\ & a_i \leq (y_i - r_i^r)/(r_i^a - r_i^r) & \forall i \\ & a_i \leq \alpha(y_i - r_i^a)/(r_i^a - r_i^r) + 1 & \forall i \end{aligned} \quad (16)$$

However, for some special sequences of the OWA weights v_i this solution concept can easily be defined without any need to order outcomes thus the solution procedure may be quite simple. From the properties of the Gini's mean absolute difference [15] it follows that

$$\sum_{i=1}^m \sum_{k=1}^m \min\{a_i, a_k\} = \sum_{i=1}^m a_i - \frac{1}{2} \sum_{i=1}^m \sum_{k=1}^m |a_i - a_k| = \sum_{i=1}^m [2(m-i) + 1] \theta_i(\mathbf{a})$$

Hence, the OWA aggregation given by the decreasing sequence of weights with a constant step $v_i - v_{i+1} = \Delta$ can be directly expressed as with the use of pairwise minimum or pairwise absolute differences which are piecewise linear functions. This leads us to the following extension of the analytic RPM model (2)

$$\max \left\{ \min_{1 \leq i \leq m} a_i + \frac{\varepsilon}{m^2} \sum_{i=1}^m \sum_{k=1}^m \min\{a_i, a_k\} : a_i = s_i(f_i(\mathbf{x})) \forall i, \mathbf{x} \in Q \right\} \quad (17)$$

Certainly, such an analytic model is only rough approximation to the nucleolar RPM. Nevertheless, when applying (17) to our sample problem from Table 1, the solution S6 is selected. For strictly increasing and

concave partial achievement functions (6) the model can be expressed as:

$$\begin{aligned}
 \max \quad & \underline{a} + \frac{\varepsilon}{m^2} \sum_{i=1}^m \sum_{k=1}^m t_{ik} \quad \text{s.t.} \quad \mathbf{x} \in Q, y_i = f_i(\mathbf{x}) & \forall i \\
 & a_i \geq \underline{a} & \forall i \\
 & a_i \geq t_{ik}, a_k \geq t_{ik} & \forall i, k \\
 & a_i \leq \gamma(y_i - r_i^r)/(r_i^a - r_i^r) & \forall i \\
 & a_i \leq (y_i - r_i^r)/(r_i^a - r_i^r) & \forall i \\
 & a_i \leq \alpha(y_i - r_i^a)/(r_i^a - r_i^r) + 1 & \forall i
 \end{aligned} \tag{18}$$

5 The Weighted OWA Aggregation

The OWA operator allows one to model various aggregation functions from the maximum through the arithmetic mean to the minimum. Thus, it enables modeling of various preferences from the optimistic to the pessimistic one. On the other hand, the OWA does not allow one to allocate any importance weights to specific scenarios. Actually, the weighted mean cannot be expressed in terms of the OWA aggregations. Torra [27] has incorporated importance weighting into the OWA operator within the Weighted OWA (WOWA) aggregation introduced as a particular case of Choquet integral using a distorted probability as the measure. The WOWA average becomes the weighted mean in the case of equal all the preference weights and it is reduced to the standard OWA average for equal all the importance weights. Since its introduction, the WOWA operator has been successfully applied to many fields of decision making [29] including metadata aggregation problems [2].

Let $\mathbf{w} = (w_1, \dots, w_m)$ be an m -dimensional vector of preferential weights such that $w_i \geq 0$ for $i = 1, 2, \dots, m$ and $\sum_{i=1}^m w_i = 1$. Further, let $\mathbf{p} = (p_1, \dots, p_m)$ be an m -dimensional vector of importance weights such that $p_i \geq 0$ for $i = 1, 2, \dots, m$ and $\sum_{i=1}^m p_i = 1$. The corresponding Weighted OWA aggregation of vector $\mathbf{a} = (a_1, \dots, a_m)$ is defined [27] as follows:

$$A_{\mathbf{w}, \mathbf{p}}(\mathbf{a}) = \sum_{i=1}^m \omega_i a_{\langle i \rangle} = \sum_{i=1}^m \omega_i \theta_{m-i+1}(\mathbf{y}) \quad \text{with} \quad \omega_i = w^* \left(\sum_{k \leq i} p_{\tau(k)} \right) - w^* \left(\sum_{k < i} p_{\tau(k)} \right) \tag{19}$$

where w^* is an increasing function interpolating points $(\frac{i}{m}, \sum_{k \leq i} w_k)$ together with the point (0.0) and τ representing the ordering permutation for $\mathbf{y}$ (i.e. $y_{\tau(i)} = \theta_i(\mathbf{y})$). Moreover, function w^* is required to be a straight line when the point can be interpolated in this way. We will focus our analysis on the piecewise linear interpolation function w^* which is the simplest form of the required interpolation.

The WOWA aggregation covers the standard weighted mean with weights p_i as a special case of equal preference weights ($w_i = 1/m$ for $i = 1, 2, \dots, m$). Actually, the WOWA operator is a particular case of Choquet integral using a distorted probability as the measure [5].

Example 1 Consider achievements vectors $\mathbf{a}' = (1, 3, 2, 4, 5)$ and $\mathbf{a}'' = (1, 1, 2, 6, 4)$. While introducing preferential weights $\mathbf{w} = (0.05, 0.1, 0.15, 0.2, 0.5)$ one may calculate the OWA averages: $A_{\mathbf{w}}(\mathbf{y}') = 0.05 \cdot 5 + 0.1 \cdot 4 + 0.15 \cdot 3 + 0.2 \cdot 2 + 0.5 \cdot 1 = 2$ and $A_{\mathbf{w}}(\mathbf{y}'') = 0.05 \cdot 6 + 0.1 \cdot 4 + 0.15 \cdot 2 + 0.2 \cdot 1 + 0.5 \cdot 1 = 1.7$. Further, let us introduce importance weights $\mathbf{p} = (0.1, 0.1, 0.2, 0.5, 0.1)$ which mean that results under the third achievement are 2 times more important than those related to criteria 1, 2 or 5, while the results under achievement 4 are even 5 times more important. To take into account the importance weights in

the WOWA aggregation (19) we introduce piecewise linear function

$$w^*(\xi) = \begin{cases} 0.05\xi/0.2 & \text{for } 0 \leq \xi \leq 0.2 \\ 0.05 + 0.10(\xi - 0.2)/0.2 & \text{for } 0.2 < \xi \leq 0.4 \\ 0.15 + 0.15(\xi - 0.4)/0.2 & \text{for } 0.4 < \xi \leq 0.6 \\ 0.3 + 0.2(\xi - 0.6)/0.2 & \text{for } 0.6 < \xi \leq 0.8 \\ 0.5 + 0.5(\xi - 0.8)/0.2 & \text{for } 0.8 < \xi \leq 1.0 \end{cases}$$

and calculate weights ω_i according to formula (19) as w^* increments corresponding to importance weights of the ordered outcomes, as illustrated in Fig. 3. In particular, one get $\omega_1 = w^*(p_5) = 0.025$ and $\omega_2 = w^*(p_5 + p_4) - w^*(p_5) = 0.275$ for vector $\mathbf{y}'$ while $\omega_1 = w^*(p_4) = 0.225$ and $\omega_2 = w^*(p_4 + p_5) - w^*(p_4) = 0.075$ for vector $\mathbf{y}''$. Finally, $A_{\mathbf{w},\mathbf{p}}(\mathbf{a}') = 0.025 \cdot 5 + 0.275 \cdot 4 + 0.1 \cdot 3 + 0.35 \cdot 2 + 0.25 \cdot 1 = 2.475$ and $A_{\mathbf{w},\mathbf{p}}(\mathbf{a}'') = 0.225 \cdot 6 + 0.075 \cdot 4 + 0.2 \cdot 2 + 0.25 \cdot 1 + 0.25 \cdot 1 = 2.55$.

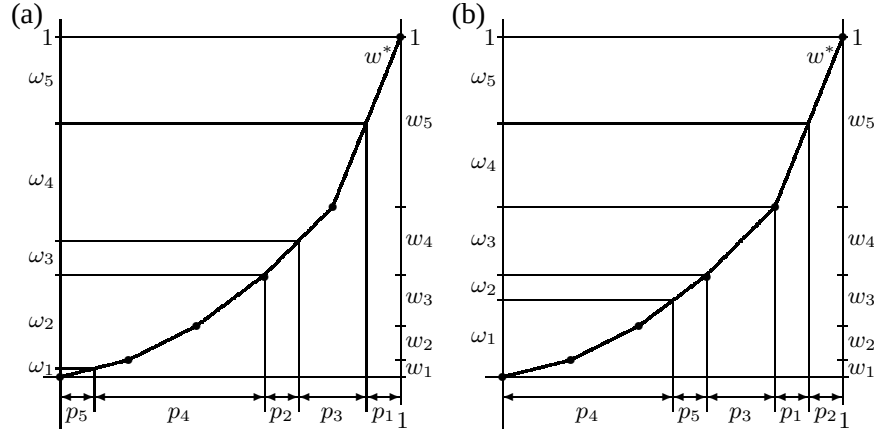


Figure 3: Definition of weights ω_i for Example 1: (a) vector $\mathbf{a}'$, (b) vector $\mathbf{a}''$

Note that one may alternatively compute the WOWA values by using the importance weights to replicate corresponding scenarios and calculate then OWA aggregations. In the case of our importance weights $\mathbf{p}$ we need to consider five copies of scenario 4 and two copies of scenario 3 thus generating corresponding vectors $\tilde{\mathbf{a}}' = (1, 3, 2, 2, 4, 4, 4, 4, 4, 5)$ and $\tilde{\mathbf{a}}'' = (1, 1, 2, 2, 6, 6, 6, 6, 6, 4)$ of ten equally important outcomes. Original five preferential weights must be then applied respectively to the average of the two largest outcomes, the average of the next two largest outcomes etc. Indeed, we get $A_{\mathbf{w},\mathbf{p}}(\mathbf{a}') = 0.05 \cdot 4.5 + 0.1 \cdot 4 + 0.15 \cdot 4 + 0.2 \cdot 2.5 + 0.5 \cdot 1.5 = 2.475$ and $A_{\mathbf{w},\mathbf{p}}(\mathbf{a}'') = 0.05 \cdot 6 + 0.1 \cdot 6 + 0.15 \cdot 5 + 0.2 \cdot 2 + 0.5 \cdot 1 = 2.55$. We will further formalize this approach and take its advantages to build LP computational models. $\square$

Function w^* can be defined by its generation function g with the formula $w^*(\alpha) = \int_0^\alpha g(\xi) d\xi$. Introducing breakpoints $\beta_i = \sum_{k \leq i} p_{\tau(k)}$ and $\beta_0 = 0$ allows us to express $\omega_i = \int_0^{\beta_i} g(\xi) d\xi - \int_0^{\beta_{i-1}} g(\xi) d\xi = \int_{\beta_{i-1}}^{\beta_i} g(\xi) d\xi$ and the entire WOWA aggregation as

$$A_{\mathbf{w},\mathbf{p}}(\mathbf{a}) = \sum_{i=1}^m a_{\langle i \rangle} \int_{\beta_{i-1}}^{\beta_i} g(\xi) d\xi = \int_0^1 g(\xi) \overline{F}_{\mathbf{a}}^{(-1)}(\xi) d\xi \quad (20)$$

where $\overline{F}_{\mathbf{a}}^{(-1)}$ is the stepwise function $\overline{F}_{\mathbf{a}}^{(-1)}(\xi) = \theta_i(\mathbf{a})$ for $\beta_{i-1} < \xi \leq \beta_i$. It can also be mathematically

formalized as follows. First, we introduce the right-continuous cumulative distribution function (cdf):

$$F_{\mathbf{y}}(d) = \sum_{i=1}^m p_i \delta_i(d) \quad \text{where} \quad \delta_i(d) = \begin{cases} 1 & \text{if } a_i \leq d \\ 0 & \text{otherwise} \end{cases} \quad (21)$$

which for any real (outcome) value d provides the measure of achievement values smaller or equal to d . Next, we introduce the quantile function $F_{\mathbf{a}}^{(-1)} = \inf \{ \eta : F_{\mathbf{a}}(\eta) \geq \xi \}$ for $0 < \xi \leq 1$ as the left-continuous inverse of the cumulative distribution function $F_{\mathbf{a}}$, ie., $F_{\mathbf{a}}^{(-1)}(\xi) = \inf \{ \eta : F_{\mathbf{a}}(\eta) \geq \xi \}$ for $0 < \xi \leq 1$, and finally $\overline{F}_{\mathbf{a}}^{(-1)}(\xi) = F_{\mathbf{a}}^{(-1)}(1 - \xi)$.

Formula (20) provides the most general expression of the WOWA aggregation allowing for expansion to continuous case. The original definition of WOWA allows one to build various interpolation functions w^* [28] thus to use different generation functions g in formula (20). We have focused our analysis on the the piecewise linear interpolation function w^* . Note, however, that the piecewise linear functions may be built with various number of breakpoints, not necessarily m . Thus, any nonlinear function can be well approximated by an piecewise linear function with appropriate number of breakpoints. Therefore, we will consider weights vectors $\mathbf{w}$ of dimension n not necessarily equal to m . Any such piecewise linear interpolation function w^* can be expressed with the stepwise generation function $g(\xi) = nw_k$ for $(k-1)/n < \xi \leq k/n$, $k = 1, 2, \dots, n$. This leads us to the following specification of formula (20):

$$A_{\mathbf{w}, \mathbf{p}}(\mathbf{y}) = \sum_{k=1}^n w_k n \int_{(k-1)/n}^{k/n} \overline{F}_{\mathbf{y}}^{(-1)}(\xi) d\xi = \sum_{k=1}^n w_k n \int_{(k-1)/n}^{k/n} F_{\mathbf{y}}^{(-1)}(1 - \xi) d\xi \quad (22)$$

Note that $n \int_{(k-1)/n}^{k/n} \overline{F}_{\mathbf{y}}^{(-1)}(\xi) d\xi$ represents the average within the k -th portion of $1/n$ largest outcomes, the corresponding conditional mean [21, 25]. Hence, formula (22) defines WOWA aggregations with preferential weights $\mathbf{w}$ as the corresponding OWA aggregation but applied to the conditional means calculated according to the importance weights $\mathbf{p}$ instead of the original outcomes. Fig. 4 illustrates application of formula (22) for computation of the WOWA aggregations in Example 1.

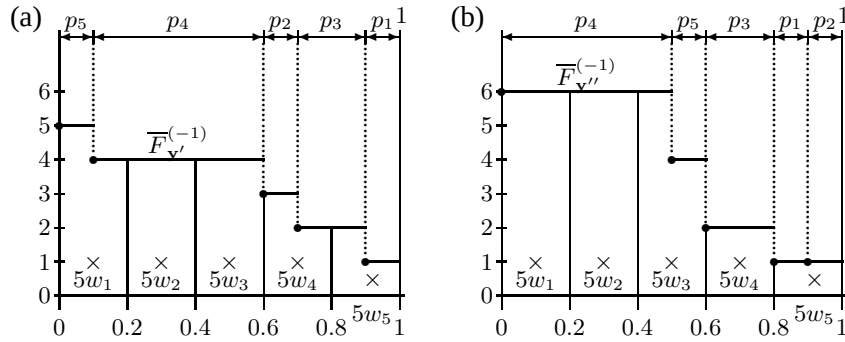


Figure 4: Formula (22) applied to calculations in Example 1: (a) vector $\mathbf{a}'$, (b) vector $\mathbf{a}''$

We will treat formula (22) as a formal definition of the WOWA aggregation of m -dimensional outcomes $\mathbf{a}$ defined by m -dimensional importance weights $\mathbf{p}$ and n -dimensional preferential weights $\mathbf{w}$. We will focus our analysis on the WOWA aggregation defined by increasing weights $w_1 \leq w_2 \leq \dots \leq w_n$. Recall that this represents decreasing weights v_i corresponding to $\theta_i(\mathbf{a})$, i.e. equally important unit of a smaller outcome is considered with a larger weight. This is mathematically represented by the convexity of function w^* as well as it may be viewed as andness of the WOWA operator [8] when considered as the OWA defined via the regular increasing monotone (RIM) quantifiers [34] ($\int_0^1 w^*(\xi) d\xi \leq 0.5$).

6 WOWA Enhanced RPM

The WOWA aggregation can be applied to the partial achievement functions in order to incorporate some importance weights for several achievements. For instance applying importance weighting $\mathbf{p} = (\frac{4}{12}, \frac{3}{12}, \frac{2}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12})$ to solution achievements from Table 3 and using them together with the OWA weights $\mathbf{w}$ from Table 2 one get the WOWA aggregations from Table 5. The corresponding RPM method selects than solution S6, similarly to the case of equal importance weights. On the other hand when increasing the importance of the last outcome achievements with $\mathbf{p} = (\frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{7}{12})$ one get the WOWA values from Table 6.

Table 5: WOWA selection

$\mathbf{w}$	0.5	0.25	0.15	0.05	0.03	0.02	$A_{\mathbf{w},\mathbf{p}}(\mathbf{a})$
S1	0.3 0.3	0.3 0.3	0.3 1.2	1.2 1.2	1.2 1.2	1.2 1.2	0.4575
S2	0.3 0.3	0.3 0.3	1.2 1.2	1.2 1.2	1.2 1.2	1.2 1.2	0.525
S3	0.3 0.3	0.3 1.2	1.2 1.2	1.2 1.2	1.2 1.2	1.2 1.2	0.6375
S4	0.3 0.3	1.2 1.2	1.2 1.2	1.2 1.2	1.2 1.2	1.2 1.2	0.75
S5	0.3 0.3	1.2 1.2	1.2 1.2	1.2 1.2	1.2 1.2	1.2 1.2	0.75
S6	0.3 1.0	1.0 1.0	1.0 1.0	1.0 1.0	1.0 1.0	1.0 1.0	0.825
S7	0.3 0.3	0.3 0.3	0.3 0.3	0.3 0.3	0.3 0.6	1.0 1.0	0.3185

Table 6: WOWA selection

$\mathbf{w}$	0.5	0.25	0.15	0.05	0.03	0.02	$A_{\mathbf{w},\mathbf{p}}(\mathbf{a})$
S1	0.3 0.3	0.3 0.3	0.3 0.3	0.3 0.3	1.2 1.2	1.2 1.2	0.345
S2	0.3 0.3	0.3 0.3	0.3 0.3	0.3 0.3	1.2 1.2	1.2 1.2	0.345
S3	0.3 0.3	0.3 0.3	0.3 0.3	0.3 0.3	1.2 1.2	1.2 1.2	0.345
S4	0.3 0.3	0.3 0.3	0.3 0.3	0.3 0.3	1.2 1.2	1.2 1.2	0.345
S5	0.3 0.3	0.3 0.3	0.3 0.3	0.3 0.3	1.2 1.2	1.2 1.2	0.345
S6	0.3 0.3	0.3 0.3	0.3 0.3	0.3 1.0	1.0 1.0	1.0 1.0	0.3525
S7	0.3 0.3	0.3 0.6	1.0 1.0	1.0 1.0	1.0 1.0	1.0 1.0	0.5125

Formula (22) defines the WOWA value applying preferential weights w_i to importance weighted averages within quantile intervals. It may be reformulated to use the tail averages

$$A_{\mathbf{w},\mathbf{p}}(\mathbf{a}) = \sum_{k=1}^m n w_k (L(\mathbf{a}, \mathbf{p}, 1 - \frac{k-1}{m}) - L(\mathbf{a}, \mathbf{p}, 1 - \frac{k}{m})) = \sum_{k=1}^m w'_k L(\mathbf{a}, \mathbf{p}, \frac{k}{m}) \quad (23)$$

where $L(\mathbf{a}, \mathbf{p}, \xi)$ is defined by left-tail integrating of $F_{\mathbf{a}}^{(-1)}$, i.e.

$$L(\mathbf{a}, \mathbf{p}, 0) = 0 \quad \text{and} \quad L(\mathbf{a}, \mathbf{p}, \xi) = \int_0^\xi F_{\mathbf{a}}^{(-1)}(\alpha) d\alpha \quad \text{for } 0 < \xi \leq 1 \quad (24)$$

while weights $w'_k = m(w_{m-k+1} - w_{m-k})$ for $k = 1, \dots, m-1$ and $w'_m = m w_1$.

Graphs of functions $L(\mathbf{a}, \mathbf{p}, \xi)$ (with respect to ξ) take the form of convex piecewise linear curves, the so-called absolute Lorenz curves [19] connected to the relation of the second order stochastic dominance

(SSD). Therefore, formula (23) relates the WOWA average to the SSD consistent risk measures based on the tail means [10] provided that the importance weights are treated as scenario probabilities.

Following (23), maximization of the WOWA aggregated achievements

$$\max\{A_{\mathbf{w},\mathbf{p}}(\mathbf{a}) : a_i = s_i(f_i(\mathbf{x})) \forall i, \mathbf{x} \in Q\} \quad (25)$$

results in problem

$$\max\left\{\sum_{k=1}^m w'_k L(\mathbf{a}, \mathbf{p}, \frac{k}{m}) : a_i = s_i(f_i(\mathbf{x})) \forall i, \mathbf{x} \in Q\right\}$$

with positive weights w'_k .

According to (24), values of function $L(\mathbf{a}, \mathbf{p}, \xi)$ for any $0 \leq \xi \leq 1$ can be given by optimization:

$$L(\mathbf{a}, \mathbf{p}, \xi) = \min_{s_i} \left\{ \sum_{i=1}^m a_i s_i : \sum_{i=1}^m s_i = \xi, \quad 0 \leq s_i \leq p_i \quad \forall i \right\} \quad (26)$$

The above problem is an LP for a given outcome vector $\mathbf{a}$ while it becomes nonlinear for $\mathbf{a}$ being a vector of variables. This difficulty can be overcome by taking advantage of the LP dual to (26). Introducing dual variable t corresponding to the equation $\sum_{i=1}^m s_i = \xi$ and variables d_i corresponding to upper bounds on s_i one gets the following LP dual expression of $L(\mathbf{a}, \mathbf{p}, \xi)$

$$L(\mathbf{a}, \mathbf{p}, \xi) = \max_{t, d_i} \left\{ \xi t - \sum_{i=1}^m p_i d_i : t - d_i \leq a_i, \quad d_i \geq 0 \quad \forall i \right\} \quad (27)$$

Therefore, maximization of the WOWA aggregation (25) can be expressed as follows

$$\begin{aligned} \max_{t_k, d_{ik}, a_i, x_j} \quad & \sum_{k=1}^n w'_k \left[\frac{k}{n} t_k - \sum_{i=1}^m p_i d_{ik} \right] \\ \text{s.t.} \quad & t_k - d_{ik} \leq a_i, \quad d_{ik} \geq 0 \quad \forall i, k \\ & a_i = s_i(f_i(\mathbf{x})) \quad \forall i, \mathbf{x} \in Q \end{aligned}$$

Taking into account piecewise linear partial achievement functions (6) one gets finally the following model for the entire WOWA Reference Point Method:

$$\begin{aligned} \max \quad & \sum_{k=1}^m w'_k z_k \\ \text{s.t.} \quad & z_k = k t_k - m \sum_{i=1}^m p_i d_{ik} \quad \forall k \\ & \mathbf{x} \in Q, \quad y_i = f_i(\mathbf{x}) \quad \forall i \\ & a_i \geq t_k - d_{ik}, \quad d_{ik} \geq 0 \quad \forall i, k \\ & a_i \leq \gamma \frac{y_i - r_i^r}{r_i^a - r_i^r} \quad \forall i \\ & a_i \leq \frac{y_i - r_i^r}{r_i^a - r_i^r} \quad \forall i \\ & a_i \leq \alpha \frac{y_i - r_i^r}{r_i^a - r_i^r} + 1 \quad \forall i \end{aligned} \quad (28)$$

7 Concluding Remarks

The reference point method is a very convenient technique for interactive analysis of the multiple criteria optimization problems. It provides the DM with a tool for an open analysis of the efficient frontier. The interactive analysis is navigated with the commonly accepted control parameters expressing reference levels for the individual objective functions. The partial achievement functions quantify the DM satisfaction from the individual outcomes with respect to the given reference levels. The final scalarizing function is built as the augmented max-min aggregation of partial achievements which means that the worst individual achievement is essentially maximized but the optimization process is additionally regularized with the term representing the average achievement. The regularization by the average achievement is easily implementable but it may disturb the basic max-min aggregation. In order to avoid inconsistencies caused by the regularization, the max-min solution may be regularized according to the Rawlsian principle of justice leading to the nucleolar RPM model. The nucleolar RPM implements a consequent max-min aggregation taking into account also the second worst achievement, the third worse and so on, thus resulting in much better modeling of the reference levels concept.

The OWA aggregation with monotonic weights combines all the partial achievements allocating the largest weight to the worst achievement, the second largest weight to the second worst achievement, the third largest weight to the third worst achievement, and so on. It approximates nucleolar RPM introducing explicit scalarizing achievement function to be interpreted as utility. Further following the concept of Weighted OWA [27] the importance weighting of several achievements may be incorporated into the RPM. Such a WOWA enhancement of the RPM uses importance weights to affect achievement importance by rescaling accordingly its measure within the distribution of achievements rather than straightforward rescaling of achievement values [26]. The ordered regularizations are more complicated in implementation due to the requirement of pointwise ordering of partial achievements. However, the recent progress in optimization methods for ordered averages [22] allows one to implement the OWA RPM quite effectively by taking advantages of piecewise linear expression of the cumulated ordered achievements. Similar model can be achieved for the WOWA RPM. Actually, in the case of concave piecewise linear partial achievement functions (typically used in the RPM), the resulting formulation extends the original constraints and criteria with simple linear inequalities thus allowing for a quite efficient implementation.

References

- [1] Behringer, F.A.: A simplex based algorithm for the lexicographically extended linear maxmin problem. *Eur. J. Opnl. Res.* **7** (1981) 274–283.
- [2] Damiani, E., De Capitani di Vimercati, S., Samarati, P., Viviani, M.: A WOWA-based aggregation technique on trust values connected to metadata. *Electronic Notes Theor. Comp. Sci.* **157** (2006) 131–142.
- [3] Ehrgott, M.: Discrete decision problems, multiple criteria optimization classes and lexicographic max-ordering. *Trends in Multicriteria Decision Making*, T.J. Stewart, R.C. van den Honert (eds.), Springer, Berlin (1998), 31–44.
- [4] Glover, F., Klingman, D.: The simplex SON method for LP/embedded network problems, *Math. Progr. Study* **15** (1981) 148–176.

- [5] Grabisch, M., Orlovski, S.A., Yager, R.R.: Fuzzy aggregation of numerical preferences. Fuzzy sets in decision analysis, operations research and statistics. Kluwer AP, Dordrecht, 1999, 31–68.
- [6] Granat, J. and Makowski, M.: ISAAP – Interactive Specification and Analysis of Aspiration-Based Preferences, European Journal of Operational Research **122** (2000), 469–485.
- [7] Lewandowski, A. and Wierzbicki, A.P.: *Aspiration Based Decision Support Systems – Theory, Software and Applications*, Berlin: Springer 1989.
- [8] Liu, X.: Some properties of the weighted OWA operator. IEEE Trans. Systems, Man and Cyber. B **368** (2006) 118–127.
- [9] Luss, H.: On equitable resource allocation problems: A lexicographic minimax approach. Oper. Res. **47** (1999) 361–378.
- [10] Mansini, R., Ogryczak, W., Speranza, M.G.: Conditional Value at Risk and Related Linear Programming Models for Portfolio Optimization, Annals of Oper. Res. **152** (2007) 227–256.
- [11] Marchi, E., Oviedo, J.A.: Lexicographic optimality in the multiple objective linear programming: the nucleolar solution. Eur. J. Opnl. Res. **57** (1992) 355–359.
- [12] Miettinen, K., Mäkelä, M.M.: On Scalarizing Functions in Multiobjective Optimization, OR Spectrum **24** (2002) 193–213.
- [13] Ogryczak, W.: On the lexicographic minimax approach to location problems. Eur. J. Opnl. Res. **100** (1997) 566–585.
- [14] Ogryczak, W.: Preemptive reference point method. In: Climaco J (ed). *Multicriteria Analysis — Proceedings of the XIth International Conference on MCDM*. Springer: Berlin 1997, 156–167.
- [15] Ogryczak, W.: Multiple Criteria Linear Programming Model for Portfolio Selection, Annals of Operations Research **97** (2000) 143–162.
- [16] Ogryczak, W.: On Goal Programming Formulations of the Reference Point Method, Journal of the Operational Research Society **52** (2001) 691–698.
- [17] Ogryczak, W.: Multiple criteria optimization and decisions under risk. Control & Cyber. **31** (2002) 975–1003.
- [18] Ogryczak, W., Lahoda, S.: Aspiration/Reservation Decision Support – A Step Beyond Goal Programming, Journal of Multi-Criteria Decision Analysis **1** (1992) 101–117.
- [19] Ogryczak, W., Ruszczyński, A.: Dual stochastic dominance and related mean-risk models, SIAM J. Optimization **13** (2002) 60–78.
- [20] Ogryczak, W., Studziński, K., Zorychta, K.: DINAS: A Computer-Assisted Analysis System for Multiobjective Transshipment Problems with Facility Location, Computers and Operations Research **19** (1992) 637–647.
- [21] Ogryczak, W., Śliwiński, T.: On equitable approaches to resource allocation problems: the conditional minimax solution, J. Telecom. Info. Tech. **3/02** (2002) 40–48.

- [22] Ogryczak W., Śliwiński, T.: On solving linear programs with the ordered weighted averaging objective. *Eur. J. Opnl. Res.* **148** (2003) 80–91.
- [23] Ogryczak W., Śliwiński, T.: On Optimization of the Importance Weighted OWA Aggregation of Multiple Criteria, ICCSA 2007, Lecture Notes in Computer Science **4705** (2007) 804–817.
- [24] Ogryczak, W., Tamir, A.: Minimizing the sum of the k largest functions in linear time. *Inform. Proc. Let.* **85** (2003) 117–122.
- [25] Ogryczak, W., Zawadzki, M.: Conditional Median — A Parametric Solution Concept for Location Problems. *Annals of Oper. Res.* **110** (2002), 167–181.
- [26] Ruiz F., Luque M., Cabello J.M.: The role of the weights in reference point based interactive methods, DSTIS 2007, Warsaw, Poland, 2007.
- [27] Torra, V.: The weighted OWA operator. *Int. J. Intell. Syst.* **12** (1997) 153–166.
- [28] Torra, V.: The WOWA operator and the interpolation function W^* : Chen and Otto's interpolation method revisited. *Fuzzy Sets Syst.* **113** (2000) 389–396.
- [29] Valls, A., Torra, V.: Using classification as an aggregation tool for MCDM. *Fuzzy Sets Syst.* **115** (2000) 159–168.
- [30] Wierzbicki, A.P.: A Mathematical Basis for Satisficing Decision Making, *Mathematical Modelling* **3** (1982) 391–405.
- [31] Wierzbicki, A.P., Makowski, M., Wessels, J.: *Model Based Decision Support Methodology with Environmental Applications*. Dordrecht: Kluwer 2000.
- [32] Yager, R.R.: On ordered weighted averaging aggregation operators in multicriteria decision making. *IEEE Trans. Systems, Man and Cyber.* **18** (1988) 183–190.
- [33] Yager, R.R.: Constrained OWA aggregation, *Fuzzy Sets Syst.* **81** (1996) 89–101.
- [34] Yager, R.R.: Quantifier guided aggregation using OWA operators. *Int. J. Intell. Syst.* **11** (1996) 49–73.
- [35] Yager, R.R.: On the analytic representation of the Leximin ordering and its application to flexible constraint propagation. *Eur. J. Opnl. Res.* **102** (1997) 176–192.
- [36] Yager, R.R., Filev, D.P.: *Essentials of Fuzzy Modeling and Control*, Wiley, New York, 1994.
- [37] Yager, R.R., Kacprzyk, J.: *The Ordered Weighted Averaging Operators: Theory and Applications*, Kluwer AP, Dordrecht, 1997.