ELSEVIER

European Journal of Operational Research 116 (1999) 33–50

EUROPEAN JOURNAL OF OPERATIONAL RESEARCH

Theory and Methodology

From stochastic dominance to mean-risk models: Semideviations as risk measures [1]

Włodzimierz Ogryczak [a,*], Andrzej Ruszczyński [b]

[a] Department of Mathematics and Computer Science, Warsaw University, Banacha 2, 02 097 Warsaw, Poland

[b] Department of Management Science and Information Systems, Rutgers University, 94 Rockafeller Rd, Piscataway, NJ 08854, USA

Received 1 November 1997; accepted 1 April 1998

Abstract

Two methods are frequently used for modeling the choice among uncertain outcomes: stochastic dominance and mean-risk approaches. The former is based on an axiomatic model of risk-averse preferences but does not provide a convenient computational recipe. The latter quantifies the problem in a lucid form of two criteria with possible trade-off analysis, but cannot model all risk-averse preferences. In particular, if variance is used as a measure of risk, the resulting mean–variance (Markowitz) model is, in general, not consistent with stochastic dominance rules. This paper shows that the standard semideviation (square root of the semivariance) as the risk measure makes the mean-risk model consistent with the second degree stochastic dominance, provided that the trade-off coefficient is bounded by a certain constant. Similar results are obtained for the absolute semideviation, and for the absolute and standard deviations in the case of symmetric or bounded distributions. In the analysis we use a new tool, the Outcome–Risk (O–R) diagram, which appears to be particularly useful for comparing uncertain outcomes.

Keywords: Decisions under risk; Portfolio optimization; Stochastic dominance; Mean-risk model

1. Introduction

Comparing uncertain outcomes is one of fundamental interests of decision theory. Our objective is to analyze relations between the existing approaches and to provide some tools to facilitate the analysis.

We consider decisions with real-valued outcomes, such as return, net profit or number of lives saved. A leading example, originating from finance, is the problem of choice among investment opportunities or portfolios having uncertain returns. Although we discuss the consequences of our analysis in the portfolio selection context, we do not assume any specificity related to this or

* Corresponding author. E-mail: wlodzimierz.ogryczak@mimuw.edu.pl

[1] This work was partially supported by the International Institute for Applied Systems Analysis, Laxenburg, Austria, and distributed on-line as the IIASA Report IR-97-027 (June 1997).

PII: S 0 3 7 7 - 2 2 1 7 (9 8) 0 0 1 6 7 - 2

another application. We consider the general problem of comparing real-valued random variables (distributions), assuming that larger outcomes are preferred. We describe a random variable X by the probability measure P_X induced by it on the real line R. It is a general framework: the random variables considered may be discrete, continuous, or mixed (Pratt et al., 1995). Owing to that, our analysis covers a variety of problems of choosing among uncertain prospects that occur in economics and management.

Two methods are frequently used for modeling choice among uncertain prospects: *stochastic dominance* (Whitmore and Findlay, 1978; Levy, 1992), and *mean-risk analysis* (Markowitz, 1987). The former is based on an axiomatic model of risk-averse preferences: it leads to conclusions which are consistent with the axioms. Unfortunately, the stochastic dominance approach does not provide us with a simple computational recipe – it is, in fact, a multiple criteria model with a continuum of criteria. The mean-risk approach quantifies the problem in a lucid form of only two criteria: the mean, representing the expected outcome, and the risk: a scalar measure of the variability of outcomes. The mean-risk model is appealing to decision makers and allows a simple trade-off analysis, analytical or geometrical. On the other hand, mean-risk approaches are not capable of modeling the entire gamut of risk-averse preferences. Moreover, for typical dispersion statistics used as risk measures, the mean-risk approach may lead to inferior conclusions.

The seminal Markowitz (1952) portfolio optimization model uses the variance as the risk measure in the mean-risk analysis. Since then many authors have pointed out that the mean–variance model is, in general, not consistent with stochastic dominance rules. The use of the semivariance rather than variance as the risk measure was already suggested by Markowitz (1959) himself. Porter (1974) showed that the mean-risk model using a fixed-target semivariance as the risk measure is consistent with stochastic dominance. This approach was extended by Fishburn (1977) to more general risk measures associated with outcomes below some fixed target. There are many arguments for the use of fixed targets. On the other hand, when one of performance measures is the expected return, the risk measure should take into account all possible outcomes *below* the mean. Therefore, we focus our analysis on central semimoments which measure the expected value of deviations below the mean. To be more precise, we consider the *absolute semideviation* (from the mean)

$$\bar{\delta}_X = \int_{-\infty}^{\mu_X} (\mu_X - \xi)\ P_X(\mathrm{d}\xi) = \frac{1}{2}\int_{-\infty}^{\infty} |\xi - \mu_X|\ P_X(\mathrm{d}\xi) \tag{1}$$

and the *standard semideviation*

$$\bar{\sigma}_X = \left(\int_{-\infty}^{\mu_X} (\mu_X - \xi)^2\ P_X(\mathrm{d}\xi) \right)^{1/2}, \tag{2}$$

where $\mu_X = E\{X\}$. We show that mean-risk models using standard or absolute semideviations as risk measures are consistent with the stochastic dominance, if a bounded set of mean-risk trade-offs is considered. In the portfolio selection context these models correspond to the Markowitz (1959, 1987) mean–semivariance model and the Konno and Yamazaki (1991) MAD model with absolute deviation.

The paper is organized as follows. In Section 3 we recall the basics of the stochastic dominance and mean-risk approaches. We also specify what we mean by consistency of these approaches. In Section 3 we introduce a convenient graphical tool for the stochastic dominance methodology: the Outcome–Risk (O–R) diagram, and we examine various risk measures within the diagram. We further use the O–R diagram to establish consistency of mean-risk models using the absolute semideviation (Section 4) and the standard semideviation (Section 5), respectively, with the second degree stochastic dominance rules. In a similar way we reexamine the standard deviation as a possible risk measure in Section 6. Owing to the use of the O–R diagram all proofs are easy, nevertheless, rigorous.

2. Stochastic dominance and mean-risk models

Stochastic dominance is based on an axiomatic model of risk-averse preferences (Fishburn, 1964). It originated in the majorization theory (Hardy et al., 1934) for the discrete case and was later extended to general distributions (Hanoch and Levy, 1969; Rothschild and Stiglitz, 1970). Since that time it has been widely used in economics and finance (see Bawa (1982) and Levy (1992) for numerous references). In the stochastic dominance approach random variables are compared by pointwise comparison of some performance functions constructed from their distribution functions.

Let X be a random variable with the probability measure P_X. The first performance function $F_X^{(1)}$ is defined as the right-continuous cumulative distribution function itself:

$$F_X^{(1)}(\eta) = F_X(\eta) = P\{X \leqslant \eta\} \quad \text{for } \eta \in R.$$

The weak relation of the first degree stochastic dominance (FSD) is defined as follows:

$$X \succeq_{\text{FSD}} Y \iff F_X(\eta) \leqslant F_Y(\eta) \quad \text{for all } \eta \in R. \tag{3}$$

The second performance function $F_X^{(2)}$ is given by areas below the distribution function F_X:

$$F_X^{(2)}(\eta) = \int_{-\infty}^{\eta} F_X(\xi)\,\mathrm{d}\xi \quad \text{for } \eta \in R,$$

and defines the weak relation of the second degree stochastic dominance (SSD):

$$X \succeq_{\text{SSD}} Y \iff F_X^{(2)}(\eta) \leqslant F_Y^{(2)}(\eta) \quad \text{for all } \eta \in R. \tag{4}$$

The corresponding strict dominance relations $\succ_{\text{FSD}}$ and $\succ_{\text{SSD}}$ are defined by the standard rule

$$X \succ Y \iff X \succeq Y \quad \text{and} \quad Y \not\succeq X. \tag{5}$$

Thus, we say that X *dominates* Y *under the FSD rules* ($X \succ_{\text{FSD}} Y$), if $F_X(\eta) \leqslant F_Y(\eta)$ for all $\eta \in R$, where at least one strict inequality holds. Similarly, we say that X *dominates* Y *under the SSD rules* ($X \succ_{\text{SSD}} Y$), if $F_X^{(2)}(\eta) \leqslant F_Y^{(2)}(\eta)$ for all $\eta \in R$, with at least one inequality strict. Certainly, $X \succeq_{\text{FSD}} Y$ implies $X \succeq_{\text{SSD}} Y$ and $X \succ_{\text{FSD}} Y$ implies $X \succ_{\text{SSD}} Y$.

Note that $F_X(\eta)$ expresses the probability of underachievement for a given target value η. Thus the first degree stochastic dominance is based on the multidimensional (continuum-dimensional) objective defined by the probabilities of underachievement for all target values. The FSD is the most general relation. If $X \succ_{\text{FSD}} Y$, then X is preferred to Y within all models preferring larger outcomes, no matter how risk-averse or risk-seeking they are.

For decision making under risk most important is the second degree stochastic dominance relation, associated with the function $F_X^{(2)}$. If $X \succ_{\text{SSD}} Y$, then X is preferred to Y within all risk-averse preference models that prefer larger outcomes. It is therefore a matter of primary importance that an approach to the comparison of random outcomes be consistent with the second degree stochastic dominance relation. Our paper focuses on the consistency of mean-risk approaches with SSD.

Mean-risk approaches are based on comparing two scalar characteristics (summary statistics), the first of which – denoted μ – represents the expected outcome (reward), and the second – denoted r – is some measure of risk. The weak relation of mean-risk dominance is defined as follows:

$$X \succeq_{\mu/r} Y \iff \mu_X \geqslant \mu_Y \quad \text{and} \quad r_X \leqslant r_Y.$$

The corresponding strict dominance relation $\succ_{\mu/r}$ is defined in the standard way, as in Eq. (5). We say that X *dominates* Y *under the* μ/r *rules* ($X \succ_{\mu/r} Y$), if $\mu_X \geqslant \mu_Y$ and $r_X \leqslant r_Y$, and at least one of these inequalities is strict. Note that random variables X and Y such that $\mu_X = \mu_Y$ and $r_X = r_Y$ are indifferent under the μ/r rules ($X \succeq_{\mu/r} Y$ and $Y \succeq_{\mu/r} X$). We say then that X and Y *generate a tie in the* μ/r *model.*

An important advantage of mean-risk approaches is the possibility of a pictorial trade-off analysis. Having assumed a trade-off coefficient λ between the risk and the mean, one may directly compare real values of $\mu_X - \lambda r_X$ and $\mu_Y - \lambda r_Y$. Indeed, the following implication holds:

$$X \succeq_{\mu/r} Y \;\Rightarrow\; \mu_X - \lambda r_X \geqslant \mu_Y - \lambda r_Y \quad \text{for all } \lambda > 0.$$

We say that the trade-off approach is consistent with the mean-risk dominance.

Suppose now that the mean-risk model is consistent with the SSD model by the implication

$$X \succeq_{\mathrm{SSD}} Y \quad \Rightarrow \quad X \succeq_{\mu/r} Y.$$

Then mean-risk and trade-off approaches lead to guaranteed results:

$$X \succ_{\mu/r} Y \quad \Rightarrow \quad Y \nsucc_{\mathrm{SSD}} X,$$

$$\mu_X - \lambda r_X > \mu_Y - \lambda r_Y \text{ for some } \lambda > 0 \\ \quad \Rightarrow \quad Y \nsucc_{\mathrm{SSD}} X.$$

In other words, they cannot strictly prefer an inferior decision.

In this paper we show that some mean-risk models are consistent with the SSD model in the following sense: *there exists a positive constant α such that for all X and Y*

$$X \succeq_{\mathrm{SSD}} Y \quad \Rightarrow \quad \mu_X \geqslant \mu_Y \quad \text{and} \\ \mu_X - \alpha\, r_X \geqslant \mu_Y - \alpha\, r_Y. \tag{6}$$

In particular, for the risk measure r defined as the absolute semideviation (1) or standard semideviation (2), the constant α turns out to be equal to 1. Yitzhaki (1982) showed a similar result for the risk measure defined as the Gini's mean (absolute) difference $r_X = \Gamma_X = \frac{1}{2}\int\int|\xi - \eta|\, P_X(\mathrm{d}\xi)P_X(\mathrm{d}\eta)$.

Relation (6) directly expresses the consistency with SSD of the model using only two criteria: μ and $\mu - \alpha\, r$. Both, however, are defined by μ and r, and we have

$$\mu_X \geqslant \mu_Y \quad \text{and} \quad \mu_X - \alpha\, r_X \geqslant \mu_Y - \alpha\, r_Y \\ \quad \Rightarrow \quad \mu_X - \lambda r_X \geqslant \mu_Y - \lambda r_Y \quad \text{for } 0 < \lambda \leqslant \alpha.$$

Consequently, Eq. (6) may be interpreted as the consistency with SSD of the mean-risk model, provided that the trade-off coefficient is bounded from above by α. Namely, Eq. (6) guarantees that

$$\mu_X - \lambda r_X > \mu_Y - \lambda r_Y \quad \text{for some } 0 < \lambda \leqslant \alpha \\ \quad \Rightarrow \quad Y \nsucc_{\mathrm{SSD}} X.$$

Comparison of random variables is usually related to the problem of choice among risky alternatives in a given feasible set Q. For instance, in the simplest problem of portfolio selection (Markowitz, 1987) the feasible set of random variables is defined as all convex combinations (weighted averages with nonnegative weights totaling 1) of a given number of investment opportunities (securities). A feasible random variable $X \in Q$ is called *efficient* under the relation $\succeq$ if there is no $Y \in Q$ such that $Y \succ X$. Consistency (6) leads to the following result.

Proposition 1. *If the mean-risk model satisfies (6), then except for random variables with identical μ and r, every random variable that is maximal by $\mu - \lambda r$ with $0 < \lambda < \alpha$ is efficient under the SSD rules.*

Proof. Let $0 < \lambda < \alpha$ and $X \in Q$ be maximal by $\mu - \lambda r$. This means that $\mu_X - \lambda r_X \geqslant \mu_Y - \lambda r_Y$ for all $Y \in Q$. Suppose that there exists $Z \in Q$ such that $Z \succ_{\mathrm{SSD}} X$. Then, from Eq. (6),

$$\mu_Z \geqslant \mu_X \quad \text{and} \quad \mu_Z - \alpha r_Z \geqslant \mu_X - \alpha r_X. \tag{7}$$

Adding these inequalities multiplied by $(1 - \lambda/\alpha)$ and λ/α, respectively, we obtain

$$(1 - \lambda/\alpha)\mu_Z + (\lambda/\alpha)(\mu_Z - \alpha r_Z) \\ \geqslant (1 - \lambda/\alpha)\mu_X + (\lambda/\alpha)(\mu_X - \alpha r_X), \tag{8}$$

which after simplification reads:

$$\mu_Z - \lambda r_Z \geqslant \mu_X - \lambda r_X.$$

But X is maximal, so we must have

$$\mu_Z - \lambda r_Z = \mu_X - \lambda r_X,$$

that is, equality in Eq. (8) holds. This combined with Eq. (7) implies $\mu_Z = \mu_X$ and $r_Z = r_X$. $\square$

Proposition 1 justifies the results of the mean-risk trade-off analysis for $0 < \lambda < \alpha$. This can be extended to $\lambda = \alpha$ provided that the inequality $\mu_X - \alpha\, r_X \geqslant \mu_Y - \alpha\, r_Y$ turns into equality only in the case of $\mu_X = \mu_Y$.

Corollary 1. *If the mean-risk model satisfies (6) as well as*

$$X \succeq_{\mathrm{SSD}} Y \quad \textit{and} \quad \mu_X > \mu_Y \\ \quad \Rightarrow \quad \mu_X - \alpha\, r_X > \mu_Y - \alpha\, r_Y \tag{9}$$

then, except for random variables with identical μ *and* r*, every random variable that is maximal by* $\mu - \lambda r$ *with* $0 < \lambda \leqslant \alpha$ *is efficient under the SSD rules.*

Proof. Due to Proposition 1, we only need to prove the case of $\lambda = \alpha$. Let $X \in Q$ be maximal by $\mu - \alpha r$. Suppose that there exists $Z \in Q$ such that $Z \succ_{\mathrm{SSD}} X$. Hence, by Eq. (6), $\mu_Z \geqslant \mu_X$. If $\mu_Z > \mu_X$, then Eq. (9) yields $\mu_X - \alpha r_X < \mu_Z - \alpha r_Z$, which contradicts the maximality of X. Thus, $\mu_Z = \mu_X$ and, by Eq. (6) and the maximality of X, one has $\mu_X - \alpha r_X = \mu_Z - \alpha r_Z$. Hence, $\mu_Z = \mu_X$ and $r_Z = r_X$. □

It follows from Proposition 1 that for mean-risk models satisfying Eq. (6) the optimal solution of the problem

$$\max\{\mu_X - \lambda\, r_X \colon X \in Q\} \tag{10}$$

with $0 < \lambda < \alpha$, if it is unique, is efficient under the SSD rules. However, in the case of nonunique optimal solutions, we only know that the optimal set of Problem (10) contains a solution which is efficient under the SSD rules. The optimal set may contain, however, also some SSD-dominated solutions. Exactly, due to Proposition 1, an optimal solution $X \in Q$ can be SSD dominated only by another optimal solution $Y \in Q$ which generates a μ/r tie with X (i.e., $\mu_Y = \mu_X$ and $r_Y = r_X$). A question arises whether it is possible to additionally regularize (refine) problem (10) in order to select those optimal solutions that are efficient under the SSD rules. We resolve this question during the analysis of specific risk measures.

In many applications, especially in the portfolio selection problem, the mean-risk model is analyzed with the so-called critical line algorithm (Markowitz, 1987). This is a technique for identifying the $\succeq_{\mu/r}$ efficient frontier by parametric optimization (10) for varying $\lambda > 0$. Proposition 1 guarantees that the part of the efficient frontier (in the μ/r image space) corresponding to trade-off coefficients $0 < \lambda < \alpha$ is also efficient under the SSD rules.

3. The O–R diagram

The second degree stochastic dominance is based on the pointwise dominance of functions $F^{(2)}$. Therefore, properties of the function $F^{(2)}$ are important for the analysis of relations between the second degree stochastic dominance and the mean-risk models. The following proposition summarizes the basic properties which we use in our analysis.

Proposition 2. *If* $E\{|X|\} < \infty$*, then the function* $F_X^{(2)}(\eta)$ *is well defined for all* $\eta \in R$ *and has the following properties:*

P1. $F_X^{(2)}(\eta)$ *is continuous, convex, nonnegative and nondecreasing.*

P2. *If* $F_X(\eta^0) > 0$*, then* $F_X^{(2)}(\eta)$ *is strictly increasing for all* $\eta \geqslant \eta^0$.

P3. $$\begin{aligned} F_X^{(2)}(\eta) &= \int_{-\infty}^{\eta} (\eta - \xi)\, \mathrm{d}F_X(\xi) \\ &= \int_{-\infty}^{\eta} (\eta - \xi)\, P_X(\mathrm{d}\xi) \\ &= P\{X \leqslant \eta\} E\{\eta - X | X \leqslant \eta\}. \end{aligned}$$

P4. $\lim_{\eta \to -\infty} F_X^{(2)}(\eta) = 0$.

P5. $$\begin{aligned} &F_X^{(2)}(\eta) - (\eta - \mu_X) \\ &= \int_{\eta}^{\infty} (\xi - \eta)\, \mathrm{d}F_X(\xi) \\ &= \int_{\eta}^{\infty} (\xi - \eta)\, P_X(\mathrm{d}\xi) \\ &= P\{X \geqslant \eta\} E\{X - \eta | X \geqslant \eta\}. \end{aligned}$$

P6. $F_X^{(2)}(\eta) - (\eta - \mu_X)$ *is a continuous, convex, nonnegative and nonincreasing function of* η.

P7. $\lim_{\eta \to \infty} [F_X^{(2)}(\eta) - (\eta - \mu_X)] = 0$.

P8. *For any given* $\eta^0 \in R$

$$\begin{aligned} &F_X^{(2)}(\eta) \\ &\quad \geqslant F_X^{(2)}(\eta^0) + (\eta - \eta^0) \sup\{F_X(\xi) \mid \xi < \eta^0\} \\ &\quad \geqslant F_X^{(2)}(\eta^0) + \eta - \eta^0 \quad \textit{for all } \eta < \eta^0, \end{aligned}$$

$$\begin{aligned} &F_X^{(2)}(\eta) \\ &\quad \leqslant F_X^{(2)}(\eta^0) + (\eta - \eta^0) \sup\{F_X(\xi) \mid \xi < \eta\} \\ &\quad \leqslant F_X^{(2)}(\eta^0) + \eta - \eta^0 \quad \textit{for all } \eta > \eta^0. \end{aligned}$$

Properties P1–P4 are rather commonly known but frequently not expressed in such a rigorous form for general random variables. Properties P5–P8 seem to be less known or at least not widely used in the stochastic dominance literature. In Appendix A we give a formal proof of Proposition 2.

From now on, we assume that all random variables under consideration are integrable in the sense that $E\{|X|\} < \infty$. Therefore, we are allowed us to use all the properties P1–P8 in our analysis.

Note that, due to property P3,

$$F_X^{(2)}(\eta) = P\{X \leqslant \eta\}E\{\eta - X \mid X \leqslant \eta\}$$

thus expressing the expected shortage for each target outcome η. So, in addition to being the most general dominance relation for all risk-averse preferences, SSD is a rather intuitive multidimensional (continuum-dimensional) risk measure. Therefore, we will refer to the graph of $F_X^{(2)}$ as to the O–R diagram for the random variable X (Fig. 1).

The graph of the function $F_X^{(2)}$ has two asymptotes which intersect at the point $(\mu_X, 0)$. Specifically, the η-axis is the left asymptote (property P4) and the line $\eta - \mu_X$ is the right asymptote (property P7). In the case of a deterministic outcome ($X = \mu_X$), the graph of $F_X^{(2)}$ coincides with the asymptotes, whereas any uncertain outcome with the same expected value μ_X yields a graph above (precisely, not below) the asymptotes. Hence, the space between the curve $(\eta, F_X^{(2)}(\eta))$, $\eta \in R$, and its asymptotes represents the dispersion (and thereby the riskiness) of X in comparison to the deterministic outcome of μ_X. We shall call it the *dispersion space*. Both size and shape of the dispersion space are important for complete description of the riskiness. Nevertheless, it is quite natural to consider some size parameters as summary characteristics of riskiness.

As the simplest size parameter one may consider the maximal vertical diameter. By properties P1 and P6, it is equal to $F_X^{(2)}(\mu_X)$. Moreover, property P3 yields the following corollary.

Corollary 2. *If* $E\{|X|\} < \infty$, *then* $F_X^{(2)}(\mu_X) = \bar{\delta}_X$.

The absolute semideviation $\bar{\delta}_X$ turns out to be a linear measure of the dispersion space.

There are many arguments (see, e.g., Markowitz, 1959) that only the dispersion related to underachievements should be considered as a measure of riskiness. In such a case we should rather focus on the downside dispersion space, that is, to the left of μ_X. Note that $\bar{\delta}_X$ is the largest vertical diameter for both the entire dispersion space and the downside dispersion space. Thus $\bar{\delta}_X$ seems to be a quite reasonable linear measure of the risk related to the representation of a random variable X by its expected value μ_X. Moreover, the absolute deviation

$$\delta_X = \int_{-\infty}^{\infty} |\xi - \mu_X| \; P_X(\mathrm{d}\xi) \tag{11}$$

is symmetric in the sense that $\delta_X = 2\bar{\delta}_X$ for any (possible nonsymmetric) random variable X. Thus absolute mean δ also can be considered a linear measure of riskiness.

A better measure of the dispersion space should be given by its area. To evaluate it one needs to calculate the corresponding integrals. The following proposition gives these results.

Proposition 3. *If* $E\{X^2\} < \infty$, *then*

$$\int_{-\infty}^{\eta} F_X^{(2)}(\zeta)\,\mathrm{d}\zeta = \frac{1}{2}\int_{-\infty}^{\eta} (\eta - \xi)^2 \; P_X(\mathrm{d}\xi)$$
$$= \frac{1}{2}P\{X \leqslant \eta\}E\{(\eta - X)^2 | X \leqslant \eta\}, \tag{12}$$

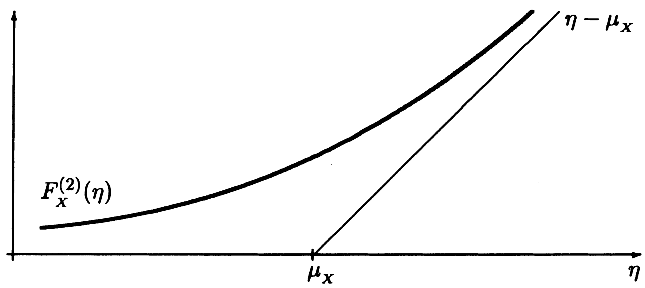

Fig. 1. The O–R diagram.

$$\int_{\eta}^{\infty} [F_X^{(2)}(\zeta) - (\zeta - \mu_X)]\, \mathrm{d}\zeta$$
$$= \frac{1}{2} \int_{\eta}^{\infty} (\xi - \eta)^2\, P_X(\mathrm{d}\xi)$$
$$= \frac{1}{2} P\{X \geqslant \eta\} E\{(X - \eta)^2 | X \geqslant \eta\}. \tag{13}$$

Formula (12) was shown by Porter (1974) for continuous random variables. The second formula seems to be new in the SSD literature. In the Appendix A we give a formal proof of both formulas for general random variables.

Corollary 3. *If* $E\{X^2\} < \infty$, *then*

$$\bar{\sigma}_X^2 = 2 \int_{-\infty}^{\mu_X} F_X^{(2)}(\zeta)\, \mathrm{d}\zeta, \tag{14}$$

$$\sigma_X^2 = 2 \int_{-\infty}^{\mu_X} F_X^{(2)}(\zeta)\, \mathrm{d}\zeta + 2 \int_{\mu_X}^{\infty} [F_X^{(2)}(\zeta) - (\zeta - \mu_X)]\, \mathrm{d}\zeta. \tag{15}$$

Hereafter, whenever considering variance σ^2 or semivariance $\bar{\sigma}^2$ (standard deviation σ or standard semideviation $\bar{\sigma}$) we will assume that $E\{X^2\} < \infty$. Therefore, we are eligible to use formulas (14) and (15) in our analysis.

By Corollary 3, the variance σ_X^2 represents the doubled area of the dispersion space of the random variable X, whereas the semivariance $\bar{\sigma}_X^2$ is the doubled area of the downside dispersion space. Thus the semimoments $\bar{\delta}$ and $\bar{\sigma}^2$, as well as the absolute moments δ and σ^2, can be regarded as some risk characteristics and they are well depicted in the O–R diagram (Figs. 2 and 3). In further sections we will use the O–R diagram to prove that the mean-risk model using the semideviations $\bar{\delta}$ and $\bar{\sigma}$ is consistent with the second degree stochastic dominance. Geometrical relations in the O–R diagram make the proofs easy. However, as the geometrical relations are the consequences of Propositions 2 and 3, the proofs are rigorous.

To conclude this section we derive some additional consequences of Propositions 2 and 3. Let us observe that in the O–R diagram the diagonal line $F_X^{(2)}(\eta^0) + \eta - \eta^0$ is parallel to the right as-

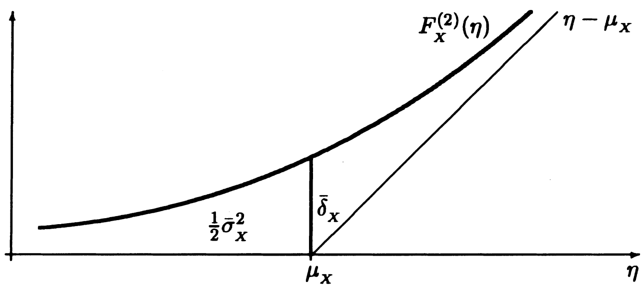

Fig. 2. The O–R diagram and the semimoments.

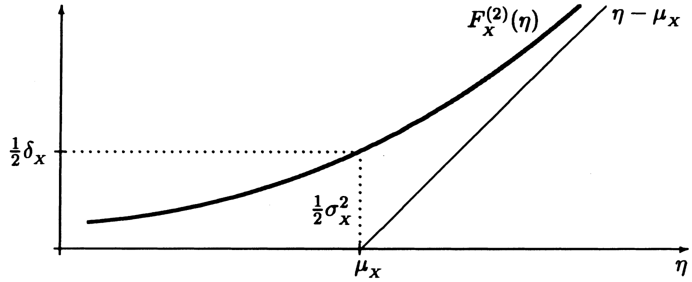

Fig. 3. The O–R diagram and the absolute moments.

ymptote $\eta - \mu_X$ and intersects the graph of $F_X^{(2)}(\eta)$ at the point $(\eta^0, F_X^{(2)}(\eta^0))$. Therefore, property P8 can be interpreted as follows. If a diagonal line (parallel to the right asymptote) intersects the graph of $F_X^{(2)}(\eta)$ at $\eta = \eta^0$, then for $\eta < \eta^0$, $F_X^{(2)}(\eta)$ is bounded from below by the line, and for $\eta > \eta^0$, $F_X^{(2)}(\eta)$ is bounded from above by the line. Moreover, the bounding is strict except in the case of $\sup\{F_X(\xi) \mid \xi < \eta^0\} = 1$ or $F_X(\eta^0) = 1$, respectively. Setting $\eta^0 = \mu_X$ we obtain the following proposition (Fig. 4).

Proposition 4. *If $E\{X^2\} < \infty$, then $\bar{\sigma}_X \geqslant \bar{\delta}_X$ and this inequality is strict except in the case $\bar{\sigma}_X = \bar{\delta}_X = 0$.*

Proof. From P8 in Proposition 2,

$$F_X^{(2)}(\eta) > F_X^{(2)}(\mu_X) + \eta - \mu_X \text{ for all } \eta < \mu_X,$$

since $\sup\{F_X(\xi) \mid \xi < \mu_X\} < 1$. Hence, in the case of $F_X^{(2)}(\mu_X) > 0$, one has $\frac{1}{2}\bar{\sigma}_X^2 > \frac{1}{2}\bar{\delta}_X^2$ and $\bar{\sigma}_X > \bar{\delta}_X$. Otherwise $\bar{\sigma}_X = \bar{\delta}_X = 0$. □

Recall that, due to the Lyapunov inequality for absolute moments (Kendall and Stuart, 1958), the standard deviation and the absolute deviation satisfy the following inequality:

$$\sigma_X \geqslant \delta_X. \tag{16}$$

Proposition 4 is its analogue for absolute and standard semideviations.

While considering two random variables X and Y in the common O–R diagram one may easily notice that, if $\mu_X < \mu_Y$, then the right asymptote of $F_X^{(2)}$ (the diagonal line $\eta - \mu_X$) must intersect the graph of $F_Y^{(2)}(\eta)$ at some η^0. By property P8,

$$F_X^{(2)}(\eta) \geqslant \eta - \mu_X \geqslant F_Y^{(2)}(\eta) \text{ for } \eta \geqslant \eta^0.$$

Moreover, since $\eta - \mu_Y$ is the right asymptote of $F_Y^{(2)}$ (property P7), there exists $\eta^1 > \eta^0$ such that

$$F_X^{(2)}(\eta) > F_Y^{(2)}(\eta) \text{ for } \eta \geqslant \eta^1.$$

Thus, from the O–R diagram one can easily derive the following, commonly known, necessary condition for the second degree stochastic dominance (Fishburn, 1980; Levy, 1992).

Proposition 5. *If $X \succeq_{\mathrm{SSD}} Y$, then $\mu_X \geqslant \mu_Y$.*

While considering in the common O–R diagram two random variables X and Y with equal expected values $\mu_X = \mu_Y$, one may easily notice that the functions $F_X^{(2)}$ and $F_Y^{(2)}$ have the same asymptotes. It leads us to the following commonly known result (Fishburn, 1980; Levy, 1992).

Proposition 6. *For random variables X and Y with equal means $\mu_X = \mu_Y$,*

$$X \succeq_{\mathrm{SSD}} Y \quad \Rightarrow \quad \sigma_X^2 \leqslant \sigma_Y^2, \tag{17}$$

$$X \succ_{\mathrm{SSD}} Y \quad \Rightarrow \quad \sigma_X^2 < \sigma_Y^2. \tag{18}$$

4. Absolute deviation as risk measure

In this section we analyze the mean-risk model with the risk defined by the absolute semideviation $\bar{\delta}$ given by Eq. (1). Recall that $\bar{\delta}_X = F_X^{(2)}(\mu_X)$ (Corollary 2) and it represents the largest vertical diameter of the (downside) dispersion space.

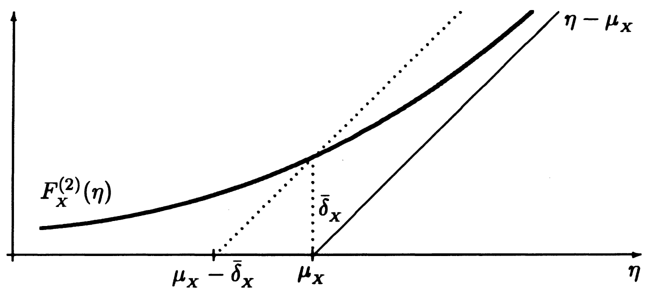

Fig. 4. $\frac{1}{2}\bar{\sigma}_X^2 \geqslant \frac{1}{2}\bar{\delta}_X^2$.

Hence, $\bar{\delta}$ is a well defined geometrical characteristic in the O–R diagram.

Consider two random variables X and Y in the common O–R diagram (Fig. 5). If $X \succeq_{\mathrm{SSD}} Y$, then, by the definition of SSD, $F_X^{(2)}$ is bounded from above by $F_Y^{(2)}$, and, by Proposition 5, $\mu_X \geqslant \mu_Y$. For $\eta \geqslant \mu_Y$, $F_Y^{(2)}(\eta)$ is bounded from above by $\bar{\delta}_Y + \eta - \mu_Y$ (second inequality of P8 in Proposition 2). Hence,

$$\bar{\delta}_X = F_X^{(2)}(\mu_X) \leqslant F_Y^{(2)}(\mu_X) \leqslant \bar{\delta}_Y + \mu_X - \mu_Y.$$

This simple analysis of the O–R diagram allows us to derive the following necessary condition for the second degree stochastic dominance.

Proposition 7. *If $X \succeq_{\mathrm{SSD}} Y$, then $\mu_X \geqslant \mu_Y$ and $\mu_X - \bar{\delta}_X \geqslant \mu_Y - \bar{\delta}_Y$, where the second inequality is strict whenever $\mu_X > \mu_Y$.*

Proof. Due to the considerations preceding the proposition, we only need to prove that $\mu_X - \bar{\delta}_X > \mu_Y - \bar{\delta}_Y$ whenever $X \succeq_{\mathrm{SSD}} Y$ and $\mu_X > \mu_Y$. Note that from the second inequality of P8 ($\eta = \mu_X$, $\eta_0 = \mu_Y$), in such a case,

$$\begin{aligned}\bar{\delta}_X &= F_X^{(2)}(\mu_X)\\ &\leqslant F_X^{(2)}(\mu_Y) + (\mu_X - \mu_Y)\ \sup\{F_X(\xi) \mid \xi < \mu_X\}\\ &< \bar{\delta}_Y + \mu_X - \mu_Y. \quad \square\end{aligned}$$

Proposition 7 says that the $\mu/\bar{\delta}$ mean-risk model is consistent with the second degree stochastic dominance by the rule (6) with $\alpha = 1$. Therefore, a $\mu/\bar{\delta}$ comparison leads to guaranteed results in the sense that

$$\mu_X - \lambda\bar{\delta}_X > \mu_Y - \lambda\bar{\delta}_Y \quad \text{for some } 0 < \lambda \leqslant 1 \\ \Rightarrow \ Y \not\succeq_{\mathrm{SSD}} X.$$

For problems of choice among risky alternatives in a given feasible set, due to Corollary 1, the following observation can be made.

Corollary 4. *Except for random variables with identical mean and absolute semideviation, every random variable $X \in Q$ that is maximal by $\mu_X - \lambda\bar{\delta}_X$ with $0 < \lambda \leqslant 1$ is efficient under the SSD rules.*

The upper bound on the trade-off coefficient λ in Corollary 4 cannot be increased for general distributions. For any $\varepsilon > 0$ there exist random variables $X \succ_{\mathrm{SSD}} Y$ such that

$$\mu_X > \mu_Y \text{ and } \mu_X - (1+\varepsilon)\bar{\delta}_X = \mu_Y - (1+\varepsilon)\bar{\delta}_Y.$$

As an example one may consider two finite random variables: X defined as $P\{X = 0\} = 1/(1+\varepsilon)$, $P\{X = 1\} = \varepsilon/(1+\varepsilon)$; and Y defined as $P\{Y = 0\} = 1$.

Konno and Yamazaki (1991) introduced the portfolio selection model based on the μ/δ mean-risk model. The model is very attractive computationally, since (for discrete random variables) it leads to linear programming problems. Therefore, it is recently applied to various finance problems (e.g., Zenios and Kang, 1993).

Note that the absolute deviation δ is a symmetric measure and the absolute semideviation $\bar{\delta}$ is its half. Hence, Proposition 7 is also valid (with factor 1/2) for the μ/δ mean-risk model. Thus, for the μ/δ model there exists a bound on the trade-offs such

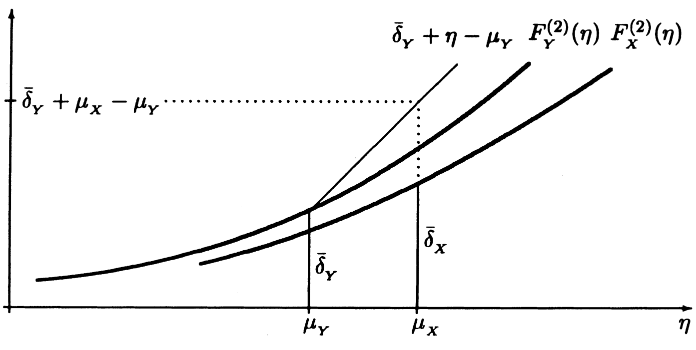

Fig. 5. SSD and the absolute semideviations: $X \succeq_{\mathrm{SSD}} Y \ \Rightarrow\ \bar{\delta}_X \leqslant \bar{\delta}_Y + \mu_X - \mu_Y$.

that for smaller trade-offs the model is consistent with the SSD rules. Specifically, due to Corollary 4, the following observation can be made.

Corollary 5. *Except for random variables with identical mean and absolute deviation, every random variable $X \in Q$ that is maximal by $\mu_X - \lambda\delta_X$ with $0 < \lambda \leqslant 1/2$ is efficient under the SSD rules.*

The upper bound on the trade-off coefficient λ in Corollary 5 can be substantially increased for symmetric distributions.

Proposition 8. *For symmetric random variables X and Y,*

$$X \succeq_{\mathrm{SSD}} Y \;\Rightarrow\; \mu_X \geqslant \mu_Y \quad \textit{and} \quad \mu_X - \delta_X \geqslant \mu_Y - \delta_Y.$$

Proof. If $X \succeq_{\mathrm{SSD}} Y$ then, due to Proposition 5, $\mu_X \geqslant \mu_Y$. From the second inequality of P8 in Proposition 2,

$$\begin{aligned}\frac{1}{2}\delta_X &= F_X^{(2)}(\mu_X)\\ &\leqslant F_X^{(2)}(\mu_Y) + (\mu_X - \mu_Y)\sup\{F_X(\xi) \mid \xi < \mu_X\}\\ &\leqslant \frac{1}{2}\delta_Y + \frac{1}{2}(\mu_X - \mu_Y),\end{aligned}$$

since for symmetric random variables

$$\sup\{F_X(\xi) \mid \xi < \mu_X\} \leqslant 1/2.$$

Hence,

$$\mu_X - \delta_X \geqslant \mu_Y - \delta_Y. \quad \square$$

For problems of choice among risky alternatives in a given feasible set, Propositions 1 and 8 imply the following result.

Corollary 6. *Within the class of symmetric random variables, except for random variables with identical mean and absolute deviation, every random variable $X \in Q$ that is maximal by $\mu_X - \lambda\delta_X$ with $0 < \lambda < 1$, is efficient under the SSD rules.*

The bound on the trade-off coefficient λ in Corollary 6 cannot be increased. There exist symmetric random variables $X \succ_{\mathrm{SSD}} Y$ such that $\mu_X > \mu_Y$ and $\mu_X - \delta_X = \mu_Y - \delta_Y$. As an example one may consider two finite random variables: X defined as $P\{X = 0\} = 0.5, P\{X = 4\} = 0.5$; and Y defined as $P\{Y = 0\} = 0.5$, $P\{Y = 2\} = 0.5$.

It follows from Corollary 4 that the optimal solution of the problem

$$\max\{\mu_X - \lambda\, \bar{\delta}_X\colon X \in Q\}, \quad 0 < \lambda \leqslant 1, \tag{19}$$

is efficient under the SSD rules, if it is unique. In the case of multiple optimal solutions, the optimal set of problem (19) contains a solution which is efficient under SSD rules but it may contain also some SSD dominated solutions. Exactly, due to Corollary 4, an optimal solution $X \in Q$ can be SSD dominated only by another optimal solution $Y \in Q$ which generates a $\mu/\bar{\delta}$ tie with X (i.e., $\mu_Y = \mu_X$ and $\bar{\delta}_Y = \bar{\delta}_X$). A question arises how different can the random variables be that generate a tie (are indifferent) in the $\mu/\bar{\delta}$ mean-risk model. Absolute semideviation is a linear measure of the dispersion space and therefore many different distributions may tie in the $\mu/\bar{\delta}$ comparison. Note that two random variables X and Y with the same expected value $\mu_X = \mu_Y$ are $\mu/\bar{\delta}$ indifferent if $F_X^{(2)}(\mu_X) = F_Y^{(2)}(\mu_X)$, independently of values of $F_X^{(2)}(\eta)$ and $F_Y^{(2)}(\eta)$ for all other $\eta \neq \mu_X$. Consider two finite random variables: X defined as $P\{X = -20\} = 0.5$, $P\{X = 20\} = 0.5$; and Y defined as $P\{Y = -1000\} = 0.01$, $P\{Y = 0\} = 0.98$, $P\{Y = 1000\} = 0.01$. They are $\mu/\bar{\delta}$ indifferent, because $\mu_X = \mu_Y = 0$ and $\bar{\delta}_X = \bar{\delta}_Y = 10$. Nevertheless, $X \succ_{\mathrm{SSD}} Y$ and $F_X^{(2)}(\eta) < F_Y^{(2)}(\eta)$ for all $0 < |\eta| < 1000$. As an extreme one may consider the case when $F_X^{(2)}(\eta) < F_Y^{(2)}(\eta)$ for all $\eta \in R$ except for $\eta = \mu_X = \mu_Y$, and despite that X and Y are $\mu/\bar{\delta}$ indifferent (Fig. 6). Therefore, the $\mu/\bar{\delta}$ model, although consistent with the second degree stochastic dominance for bounded trade-offs, dramatically needs some additional regularization to resolve ties in comparisons.

Ties in the $\mu/\bar{\delta}$ model can be resolved with additional comparisons of standard deviations or variances. By its definition, a tie in the $\mu/\bar{\delta}$ model may occur only in the case of equal means, and this is exactly the case when Proposition 6 applies. We can simply select from X and Y the one that has a smaller standard deviation. It can be formalized as the following lexicographic comparison:

$$(\mu_X - \lambda\bar{\delta}_X, -\sigma_X) \geqslant_{\mathrm{lex}} (\mu_Y - \lambda\bar{\delta}_Y, -\sigma_Y)$$
$$\iff \mu_X - \lambda\bar{\delta}_X > \mu_Y - \lambda\bar{\delta}_Y \text{ or}$$
$$\mu_X - \lambda\bar{\delta}_X = \mu_Y - \lambda\bar{\delta}_Y \quad \text{and} \quad -\sigma_X \geqslant -\sigma_Y.$$

The lexicographic relation defines a linear order. Hence, for problems of choice among risky alternatives in a given feasible set, the lexicographic maximization of $(\mu - \lambda\bar{\delta}, -\sigma)$ is well defined. It has two phases: the maximization of $\mu - \lambda\bar{\delta}$ within the feasible set, and the selection of the optimal solution that has the smallest standard deviation σ. Owing to Eq. (18), such a selection results in SSD efficient solutions (even in the case of multiple optimal solutions).

Corollary 7. *Every random variable $X \in Q$ that is lexicographically maximal by $(\mu_X - \lambda\bar{\delta}_X, -\sigma_X)$ with $0 < \lambda \leqslant 1$ is efficient under the SSD rules.*

For the μ/δ portfolio selection model (Konno and Yamazaki, 1991) the results of our analysis can be summarized as follows. While identifying the μ/δ efficient frontier by parametric optimization

$$\max\{\mu_X - \lambda\ \delta_X \colon X \in Q\} \tag{20}$$

for trade-off λ varying in the interval $(0, 0.5]$ the corresponding image in the μ/δ space represents SSD efficient solutions. Thus it can be used as the mean-risk map to seek a satisfactory μ/δ compromise. It does not mean, however, that the solutions generated during the parametric optimization (20) are SSD efficient. Therefore, having decided on some values of μ and δ one should apply the regularization technique (minimization of standard deviation) to select a specific portfolio which is SSD efficient.

5. Standard semideviation as risk measure

In this section we analyze the mean-risk model with the risk defined by the standard semideviation $\bar{\sigma}$ given by Eq. (2). Recall that the standard semideviation is the square root of the semivariance which equals to the doubled area of the downside dispersion space (Corollary 3). Hence, $\bar{\sigma}$ is a well defined geometrical characteristic in the O–R diagram.

Consider two random variables X and Y in the common O–R diagram (Fig. 7). If $X \succeq_{\mathrm{SSD}} Y$, then, by the definition of SSD, $F_X^{(2)}$ is bounded from above by $F_Y^{(2)}$, and, by Proposition 5, $\mu_X \geqslant \mu_Y$. Due to the convexity of $F_X^{(2)}$, the downside dispersion space of X is no greater than the downside dispersion space of Y plus the area of the trapezoid with the vertices: $(\mu_Y, 0)$, $(\mu_X, 0)$, $(\mu_X, F_X^{(2)}(\mu_X))$ and $(\mu_Y, F_Y^{(2)}(\mu_Y))$. Formally,

$$\frac{1}{2}\bar{\sigma}_X^2 \leqslant \frac{1}{2}\bar{\sigma}_Y^2 + \frac{1}{2}(\mu_X - \mu_Y)(\bar{\delta}_X + \bar{\delta}_Y). \tag{21}$$

This inequality allows us to derive new necessary conditions for the consistency with SSD of the bicriteria mean-risk model using standard semideviation as the risk measure.

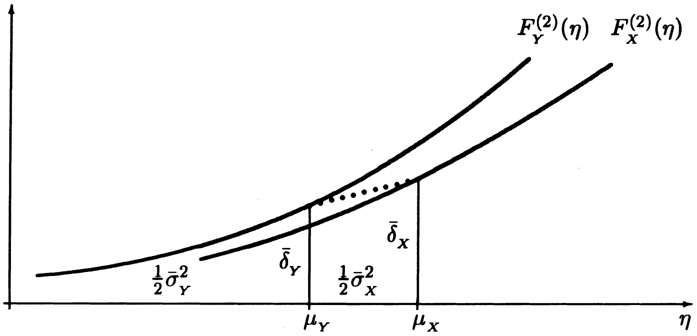

Fig. 7. SSD and the semivariances: $X \succeq_{\mathrm{SSD}} Y \Rightarrow \frac{1}{2}\bar{\sigma}_X^2 \leqslant \frac{1}{2}\bar{\sigma}_Y^2 + \frac{1}{2}(\mu_X - \mu_Y)(\bar{\delta}_X + \bar{\delta}_Y)$.

Proposition 9. *If $X \succeq_{\text{SSD}} Y$, then $\mu_X \geqslant \mu_Y$ and $\mu_X - \bar{\sigma}_X \geqslant \mu_Y - \bar{\sigma}_Y$, where the second inequality is strict whenever $\mu_X > \mu_Y$.*

Proof. If $X \succeq_{\text{SSD}} Y$ then, due to Proposition 5, $\mu_X \geqslant \mu_Y$. Moreover, inequality (21) is valid. From Proposition 4 we have $\bar{\sigma}_X \geqslant \bar{\delta}_X$ and $\bar{\sigma}_Y \geqslant \bar{\delta}_Y$. Using these inequalities in (21) we get

$$\bar{\sigma}_X^2 - \bar{\sigma}_Y^2 \leqslant (\mu_X - \mu_Y)(\bar{\sigma}_X + \bar{\sigma}_Y).$$

Hence, $\bar{\sigma}_X - \bar{\sigma}_Y \leqslant \mu_X - \mu_Y$, and finally $\mu_X - \bar{\sigma}_X \geqslant \mu_Y - \bar{\sigma}_Y$.

Moreover, from Proposition 4, $\bar{\sigma}_X = \bar{\delta}_X$ and $\bar{\sigma}_Y = \bar{\delta}_Y$ can occur only if $\bar{\sigma}_X = \bar{\sigma}_Y = 0$. Hence,

$$X \succeq_{\text{SSD}} Y \quad \text{and} \quad \mu_X > \mu_Y \quad \Rightarrow \quad \mu_X - \bar{\sigma}_X > \mu_Y - \bar{\sigma}_Y,$$

which completes the proof. □

The message of Proposition 9 is that the $\mu/\bar{\sigma}$ mean-risk model is consistent with the second degree stochastic dominance by the rule (6) with $\alpha = 1$. Therefore, $\mu/\bar{\sigma}$ comparisons lead to guaranteed results in the sense that

$$\mu_X - \lambda\bar{\sigma}_X > \mu_Y - \lambda\bar{\sigma}_Y \quad \text{for some } 0 < \lambda \leqslant 1 \quad \Rightarrow \quad Y \not\succeq_{\text{SSD}} X.$$

For problems of choice among risky alternatives in a given feasible set, Corollary 1 results in the following observation.

Corollary 8. *Except for random variables with identical mean and standard semideviation, every random variable $X \in Q$ that is maximal by $\mu_X - \lambda\bar{\sigma}_X$ with $0 < \lambda \leqslant 1$ is efficient under the SSD rules.*

The upper bound on the trade-off coefficient λ in Corollary 8 cannot be increased for general distributions. For any $\varepsilon > 0$ there exist random variables $X \succ_{\text{SSD}} Y$ such that $\mu_X > \mu_Y$ and $\mu_X - (1+\varepsilon)\bar{\sigma}_X = \mu_Y - (1+\varepsilon)\bar{\sigma}_Y$. As an example one may consider two finite random variables: X defined as $P\{X = 0\} = (1+\varepsilon)^{-2}$, $P\{X = 1\} = 1 - (1+\varepsilon)^{-2}$; and $Y = 0$.

It follows from Corollary 8 that the optimal solution of the problem

$$\max\{\mu_X - \lambda\,\bar{\sigma}_X\colon X \in Q\}, \quad 0 < \lambda \leqslant 1 \tag{22}$$

is efficient under the SSD rules, if it is unique. In the case of nonunique optimal solutions, however, we only know that the optimal set of problem (22) contains a solution which is efficient under SSD rules. Thus, similar to the $\mu/\bar{\delta}$ model, the $\mu/\bar{\sigma}$ model may generate ties (Fig. 8) and the optimal set of problem (22) may contain also some SSD dominated solutions. However, two random variables that generate a tie (are indifferent) in the $\mu/\bar{\sigma}$ mean-risk model cannot be so much different as in the $\mu/\bar{\delta}$ model. Standard semideviation $\bar{\sigma}_X$ is an area measure of the downside dispersion space and therefore it takes into account *all* values of $F_X^{(2)}(\eta)$ for $\eta \leqslant \mu_X$. Note that, if two random variables X and Y generate a tie in the $\mu/\bar{\sigma}$ model, then

$$\mu_X = \mu_Y \quad \text{and} \quad \int_{-\infty}^{\mu_X} F_X^{(2)}(\zeta)\,\mathrm{d}\zeta = \int_{-\infty}^{\mu_Y} F_Y^{(2)}(\zeta)\,\mathrm{d}\zeta.$$

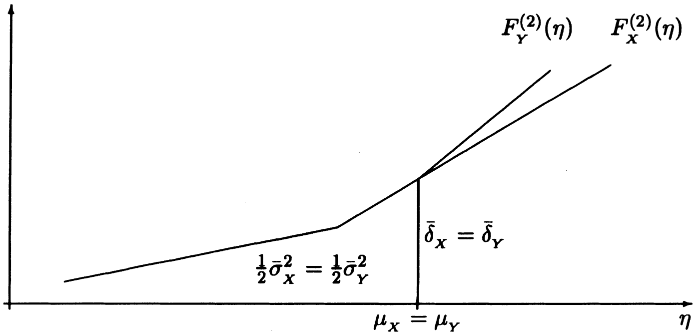

Fig. 8. A tie in the $\mu/\bar{\sigma}$ model: $\mu_X = \mu_Y$, $\bar{\sigma}_X = \bar{\sigma}_Y$ and $X \succ_{\text{SSD}} Y$.

Functions $F^{(2)}(\eta)$ are continuous and nonnegative. Hence, if $X \succeq_{\rm SSD} Y$ generate a $\mu/\bar{\sigma}$ tie, then $F_X^{(2)}(\eta) = F_Y^{(2)}(\eta)$ for all $\eta \leqslant \mu_X$. Thus a tie in the $\mu/\bar{\sigma}$ model may happen for $X \succ_{\rm SSD} Y$ but the second degree stochastic dominance X over Y is then related to overperformances rather than the underperformances. Summing up, the $\mu/\bar{\sigma}$ model needs some additional regularization to resolve ties in comparisons, but it is not such a dramatic need as in the $\mu/\bar{\delta}$ model.

Similar to the $\mu/\bar{\delta}$ model, ties in the $\mu/\bar{\sigma}$ model can be resolved by additional comparisons of standard deviations or variances. In the case when $\mu_X - \lambda\bar{\sigma}_X = \mu_Y - \lambda\bar{\sigma}_Y$, one may select from X and Y the one that has a smaller standard deviation. It can be formalized as the following lexicographic comparison:

$$\begin{aligned} &(\mu_X - \lambda\bar{\sigma}_X, -\sigma_X) \geqslant_{\rm lex} (\mu_Y - \lambda\bar{\sigma}_Y, -\sigma_Y) \\ &\quad \iff \mu_X - \lambda\bar{\sigma}_X > \mu_Y - \lambda\bar{\sigma}_Y \text{ or} \\ &\quad \mu_X - \lambda\bar{\sigma}_X = \mu_Y - \lambda\bar{\sigma}_Y \quad \text{and} \quad -\sigma_X \geqslant -\sigma_Y. \end{aligned}$$

For problems of choice among risky alternatives in a given feasible set, the lexicographic maximization of $(\mu - \lambda\bar{\sigma}, -\sigma)$ has two phases again: maximization of $\mu - \lambda\bar{\sigma}$ on the feasible set, and selection of the optimal solution that has the smallest standard deviation σ, if many optimal solutions occur. Due to Eq. (18), such a selection results in SSD efficient solutions (even in the case of multiple optimal solutions).

Corollary 9. *Every random variable $X \in Q$ that is lexicographically maximal by $(\mu_X - \lambda\bar{\sigma}_X, -\sigma_X)$ with $0 < \lambda \leqslant 1$ is efficient under the SSD rules.*

The mean–semivariance optimization approach was proposed by Markowitz (1959). It is quite an intuitive modification of the mean–variance model, since an investor worries about underperformance rather than overperformance. Nevertheless, it is less used in portfolio optimization. One reason is that it is more difficult to compute the mean–semivariance efficient frontier that for the mean–variance model. Still, Markowitz et al. (1993) have developed a critical line algorithm for the mean–semivariance efficient frontier.

The use of semivariance $\bar{\sigma}^2$ or standard semideviation $\bar{\sigma}$ in the mean-risk analysis may be considered to be equivalent, with the former easier to implement. In fact, both define exactly the same efficient set, since standard deviation is nonnegative and the square function is strictly increasing for nonnegative arguments. However, our result that the $\mu/\bar{\sigma}$ model with trade-offs bounded by one is consistent with the SSD rules cannot be directly applied to the mean–semivariance model. Note that $X \in Q$ that is maximal by $\mu - \lambda\bar{\sigma}^2$ may be not maximal by $\mu - (\lambda\bar{\sigma}_X)\bar{\sigma}$, in general. Consider two random variables X and Y with $\mu_X = 0$, $\bar{\sigma}_X = 1$ and $\mu_Y = 1$, $\bar{\sigma}_Y = 2$, respectively. For $\lambda = 0.4$, $\mu_X - 0.4\bar{\sigma}_X^2 = -0.4 > -0.6 = \mu_Y - 0.4\bar{\sigma}_Y^2$ but $\mu_X - 0.4\bar{\sigma}_X = -0.4 < 0.2 = \mu_Y - 0.4\bar{\sigma}_Y$. While comparing two random variables X and Y by the mean–semivariance trade-off analysis the following relationship is valid:

$$\begin{aligned} &\mu_X - \lambda\bar{\sigma}_X^2 \geqslant \mu_Y - \lambda\bar{\sigma}_Y^2 \iff \mu_X - [\lambda(\bar{\sigma}_X \\ &\quad + \bar{\sigma}_Y)]\bar{\sigma}_X \geqslant \mu_Y - [\lambda(\bar{\sigma}_X + \bar{\sigma}_Y)]\bar{\sigma}_Y. \end{aligned}$$

Thus for problems where the $\mu/\bar{\sigma}^2$ efficient set is bounded (like typical portfolio selection problems), there exists an upper bound on the trade-off coefficients which guarantees that for smaller trade-offs the corresponding mean-risk efficient solutions are also efficient under the SSD rules. It explains the high number of SSD efficient solutions included in the $\mu/\bar{\sigma}^2$ efficient set observed in experiments with real-life portfolio selection problems (Porter, 1974). The upper bound, though, may be very tight.

6. Standard deviation as risk measure

After the work of Markowitz (1952) the variance (or the standard deviation) is the most frequently used risk measure in mean-risk models for portfolio selection. The O–R diagram (Fig. 3) shows the variance as a natural area measure of dispersion. Comparison of random variables with equal means leads to guaranteed results (Proposition 6). However, for general random variables X and Y with *unequal* means, no relation involving their standard deviations is known to be necessary

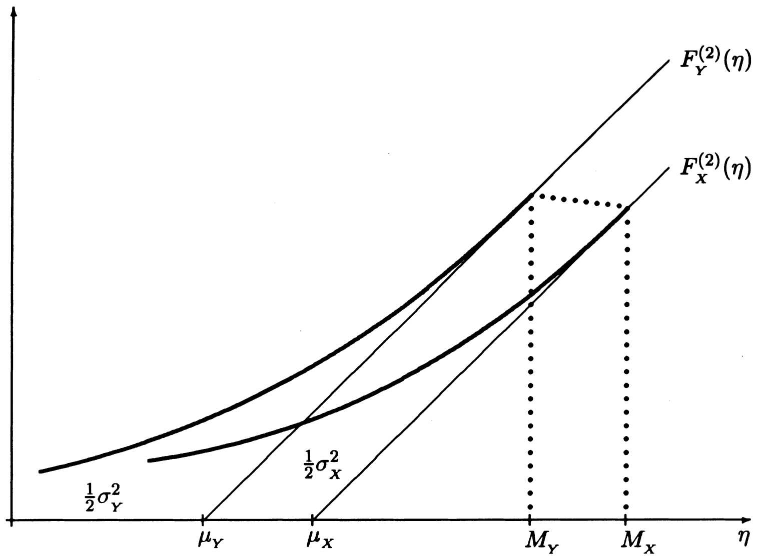

Fig. 9. SSD and the variances: $X \succeq_{\text{SSD}} Y \;\Rightarrow\; \frac{1}{2}\sigma_X^2 \leqslant \frac{1}{2}\sigma_Y^2 + \frac{1}{2}(\mu_X - \mu_Y)[(M_X - \mu_X) + (M_Y - \mu_Y)]$.

for the second degree stochastic dominance of X over Y.

In the case of symmetric distributions one has $\sigma = \sqrt{2}\bar{\sigma}$, and $\bar{\sigma}$ in Proposition 9 can be replaced with standard deviation σ multiplied by factor $\sqrt{2}/2$. It turns out, however, that for symmetric distributions the relation between $\bar{\delta}$ and $\bar{\sigma}$ can be described in more detail.

Proposition 10. *For symmetric random variables X and Y,*

$$X \succeq_{\text{SSD}} Y \;\Rightarrow\; \mu_X \geqslant \mu_Y \quad \textit{and} \quad \mu_X - \sigma_X \geqslant \mu_Y - \sigma_Y. \tag{23}$$

Proof. If $X \succeq_{\text{SSD}} Y$ then, by Proposition 5, $\mu_X \geqslant \mu_Y$. Moreover, inequality (21) is valid. Lyapunov inequality (16) for symmetric variables yields $\sigma_X \geqslant 2\bar{\delta}_X$ and $\sigma_Y \geqslant 2\bar{\delta}_Y$. Using these inequalities in (21) we get

$$\sigma_X^2 - \sigma_Y^2 \leqslant (\mu_X - \mu_Y)(\sigma_X + \sigma_Y).$$

Hence, $\sigma_X - \sigma_Y \leqslant \mu_X - \mu_Y$, and finally $\mu_X - \sigma_X \geqslant \mu_Y - \sigma_Y$ which completes the proof. □

For problems of choice among risky alternatives in a given feasible set, Propositions 1 and 6 imply the following result.

Corollary 10. *Within the class of symmetric random variables, every random variable $X \in Q$ that is maximal by $\mu_X - \lambda\sigma_X$ with $0 < \lambda < 1$, is efficient under the SSD rules.*

The bound on the trade-off coefficient in Corollary 10 is the best in the sense that there exist symmetric random variables $X \succ_{\text{SSD}} Y$ such that $\mu_X - \sigma_X = \mu_Y - \sigma_Y$. As an example one may consider two finite random variables: X defined as $P\{X = 0\} = 0.5, P\{X = 4\} = 0.5$; and Y defined as $P\{Y = 0\} = 0.5$, $P\{Y = 2\} = 0.5$. Therefore, the upper bound on the trade-off coefficient λ in Corollary 10 cannot be increased.

In the general case of nonsymmetric random variables, standard deviation is not a symmetric measure and there is no direct analogue of Proposition 9 for the standard deviation. Some similar, but much weaker, necessary conditions for the SSD dominance can be derived for distributions bounded from above (random variables with an upper bounded support). Note that, if X is upper bounded by a real number M_X (i.e. $P\{X > M_X\} = 0$), then $M_X \geqslant \mu_X$ and for $\eta \geqslant M_X$, $F_X(\eta) = 1$. Thus for $\eta \geqslant M_X$ the function $F_X^{(2)}(\eta)$ coincides with its right asymptote ($F_X^{(2)}(\eta) = \eta - \mu_X$). Hence

$$\sigma_X^2 = 2\int_{-\infty}^{\mu_X} F_X^{(2)}(\zeta)\,\mathrm{d}\zeta + 2\int_{\mu_X}^{M_X} [F_X^{(2)}(\zeta) - (\zeta - \mu_X)]\,\mathrm{d}\zeta.$$

Consider two random variables X and Y such that $P\{X > M_X\} = 0$ and $P\{Y > M_Y\} = 0$ in the common O–R diagram (Fig. 9). If $X \succeq_{\mathrm{SSD}} Y$, then, by the definition of SSD, $F_X^{(2)}$ is bounded from above by $F_Y^{(2)}$ and, by Proposition 5, one has $\mu_X \geqslant \mu_Y$. Due to the convexity of $F_X^{(2)}$, the area between this function and its asymptotes cannot be greater than the area between $F_Y^{(2)}$ and its asymptotes plus the area of the trapezoid defined by the vertices: $(\mu_Y, 0)$, $(\mu_X, 0)$, $(M_X, M_X - \mu_X)$ and $(M_Y, M_Y - \mu_Y)$. This is valid for $M_X > M_Y$ (like in Fig. 9), as well as for $M_X \leqslant M_Y$. Formally,

$$\frac{1}{2}\sigma_X^2 \leqslant \frac{1}{2}\sigma_Y^2 + \frac{1}{2}(\mu_X - \mu_Y)[(M_X - \mu_X) + (M_Y - \mu_Y)]. \tag{24}$$

This inequality is similar but stronger than the inequality derived by Levy (1992), Theorem 9, p. 570, which reads:

$$\sigma_X^2 - \sigma_Y^2 \leqslant (\mu_X - \mu_Y)(2\,\max\{M_X, M_Y\} - \mu_X - \mu_Y).$$

Inequality (24) allows for the formulation of the following necessary conditions for the bicriteria mean-risk model with the standard deviation used as the risk measure.

Proposition 11. *Suppose that a common upper bound* $h > 0$ *is known for* $(X - \mu_X)/\sigma_X$ *and* $(Y - \mu_Y)/\sigma_Y$. *Then*

$$X \succeq_{\mathrm{SSD}} Y \quad\Rightarrow\quad \mu_X \geqslant \mu_Y \text{ and } \mu_X - \frac{1}{h}\,\sigma_X \geqslant \mu_Y - \frac{1}{h}\,\sigma_Y.$$

Proof. If $X \succeq_{\mathrm{SSD}} Y$, then due to Proposition 5, $\mu_X \geqslant \mu_Y$. Further, note that inequality (24) can be rewritten as

$$\sigma_X^2 - \sigma_Y^2 \leqslant h(\sigma_X + \sigma_Y)(\mu_X - \mu_Y).$$

This immediately yields the required result. □

Corollary 11. *Within the class of random variables such that* $P\{X > \mu_X + h\sigma_X\} = 0$, *every random variable that is maximal by* $\mu_X - \lambda\sigma_X$ *with* $0 < \lambda < 1/h$, *is efficient under the SSD rules.*

Note that Proposition 11 is applicable to any pair of finite random variables. Similarly, Corollary 11 shows that in the case of a portfolio selection problem with finite random variables (for example defined by historical data), there exists a positive bound on the trade-off coefficient for the standard deviation which guarantees that for smaller trade-offs the corresponding mean-risk efficient solutions are also efficient under the SSD rules.

Analogously to the case of the semivariance discussed in the previous section, an upper bound on the variance trade-off coefficients exists which guarantees that μ/σ^2 efficient solutions are also efficient under the SSD rules. However, the upper bound may be very tight. Corollary 11 provides a theoretical explanation for the results of numerous experimental comparisons of mean–variance and SSD efficient sets on real-life portfolio selection problems (Porter, 1974, and references therein). Most of them, like that performed by Porter and Gaumnitz (1972) on over 900 portfolios of securities randomly selected from the Chicago Price Relative File, provided some support for the idea that mean–variance and SSD choices are empirically similar. The main difference was the tendency of the mean–variance efficient set to include some low mean, low variance portfolios that were eliminated by the SSD rules. Although efficient in the mean–variance analysis, they obviously correspond to large trade-off coefficients for the variance.

7. Concluding remarks

The second degree stochastic dominance relation is based on an axiomatic model of risk-averse preferences, but does not provide us with a simple computational recipe.

The mean-risk approach quantifies the problem in only two criteria: the mean, representing the expected outcome, and the risk: a scalar measure of the variability of outcomes (usually, a central moment or the corresponding deviation). This is appealing to decision makers and allows a simple trade-off analysis, analytical or geometrical.

In the paper we have analyzed the consistency of these two approaches. We have shown that standard semideviation (the square root of the semivariance) as the risk measure makes the mean-risk model consistent with the second degree stochastic dominance, provided that the trade-off coefficient is bounded by one. Similar results have been obtained for the absolute semideviation as the risk measure. These results are valid for all (possibly nonsymmetric) random variables for which these moments are well-defined. In the case of symmetric random variables the same results are valid for the standard and absolute deviations, respectively.

In many applications, especially in portfolio selection problems, the mean-risk model is analyzed by the critical line algorithm. This is a technique for identifying the mean-risk efficient frontier via parametric optimization with a varying trade-off coefficient. Our results guarantee that when risk is measured by the standard or the absolute semideviation (the standard or the absolute deviation in the case of symmetric distributions), the part of the efficient frontier (in the mean-risk image space) corresponding to trade-off coefficients smaller than one is also efficient under the SSD rules. In some way our analysis justifies the critical line methodology for typical risk measures, provided that it is not extended too far in terms of the trade-off coefficient. It also explains some results of experimental comparisons of the SSD and mean-risk efficient sets for portfolio selection problems.

In the analysis we have used a new graphical tool, the O–R diagram, which appears to be particularly convenient for comparing uncertain outcomes and examining second degree stochastic dominance. Typical dispersion statistics, commonly used as risk measures (absolute deviation and semideviation, variance and semivariance) are well depicted in the O–R diagram, and it may be useful for various types of comparisons of uncertain outcomes, especially in computerized decision support systems.

Appendix A

A.1. Proof of Proposition 2

P1. Simple consequence from definition of $F_X^{(2)}$.

P2. Simple consequence from definition of $F_X^{(2)}$.

P3. Changing the order of integration by Fubini's theorem we obtain

$$
\begin{aligned}
F_X^{(2)}(\eta) &= \int_{-\infty}^{\eta} F_X(\zeta)\,\mathrm{d}\zeta \\
&= \int_{-\infty}^{\eta} \int_{-\infty}^{\zeta} P_X(\mathrm{d}\xi)\,\mathrm{d}\zeta \\
&= \int_{-\infty}^{\eta} \int_{\xi}^{\eta} \mathrm{d}\zeta\; P_X(\mathrm{d}\xi) \\
&= \int_{-\infty}^{\eta} (\eta-\xi)\; P_X(\mathrm{d}\xi) \\
&= P\{X \leqslant \eta\} E\{\eta - X | X \leqslant \eta\}.
\end{aligned}
$$

P4.

$$
\begin{aligned}
0 &\leqslant \lim_{\eta\to-\infty} F_X^{(2)}(\eta) = \lim_{\eta\to-\infty} \int_{-\infty}^{\eta} (\eta-\xi)\; P_X(\mathrm{d}\xi) \\
&\leqslant \lim_{\eta\to-\infty} \int_{-\infty}^{\eta} |\xi|\; P_X(\mathrm{d}\xi) = 0,
\end{aligned}
$$

because $\eta - \xi \leqslant |\xi|$ for $\xi \leqslant \eta < 0$, and $E\{|X|\} < \infty$.

P5.

$$
\begin{aligned}
&F_X^{(2)}(\eta) - (\eta - \mu_X) \\
&\quad = \int_{-\infty}^{\eta} (\eta-\xi)\; P_X(\mathrm{d}\xi) - \eta + \mu_X \\
&\quad = \int_{-\infty}^{\eta} \eta\; P_X(\mathrm{d}\xi) + \int_{\eta}^{\infty} \xi\; P_X(\mathrm{d}\xi) - \eta \\
&\quad = \int_{\eta}^{\infty} (\xi-\eta)\; P_X(\mathrm{d}\xi) \\
&\quad = P\{X \geqslant \eta\} E\{X - \eta | X \geqslant \eta\}.
\end{aligned}
$$

P6. Simple consequence from P5.

P7.

$$0 \leqslant \lim_{\eta\to\infty}[F_X^{(2)}(\eta) - (\eta - \mu_X)]$$
$$= \lim_{\eta\to\infty}\int_\eta^\infty (\xi - \eta)\ P_X(\mathrm{d}\xi)$$
$$\leqslant \lim_{\eta\to\infty}\int_\eta^\infty \xi\ P_X(\mathrm{d}\xi) = 0,$$

similarly to P4.
P8. If $\eta < \eta^0$, then

$$F_X^{(2)}(\eta^0) - F_X^{(2)}(\eta) = \int_\eta^{\eta^0} F_X(\xi)\mathrm{d}\xi \leqslant (\eta^0 - \eta)\ \sup\{F_X(\xi)\ |\ \xi < \eta^0\} \leqslant \eta^0 - \eta.$$

Interchanging η and η^0 we obtain the second inequality.

A.2. Proof of Proposition 3

By Fubini's theorem,

$$\int_{-\infty}^{\eta} F_X^{(2)}(\zeta)\ \mathrm{d}\zeta = \int_{-\infty}^{\eta}\left[\int_{-\infty}^{\zeta}(\zeta-\xi)P_X(\mathrm{d}\xi)\right]\mathrm{d}\zeta$$
$$= \int\int_{\substack{\zeta\leqslant\eta\\ \xi\leqslant\zeta}} (\zeta - \xi)\ P_X(\mathrm{d}\xi)\ \mathrm{d}\zeta$$
$$= \int_{-\infty}^{\eta}\left[\int_{\xi}^{\eta}(\zeta - \xi)\ \mathrm{d}\zeta\right] P_X(\mathrm{d}\xi)$$
$$= \frac{1}{2}\int_{-\infty}^{\eta}(\eta - \xi)^2\ P_X(\mathrm{d}\xi)$$
$$= \frac{1}{2}P\{X \leqslant \eta\}E\{(\eta - X)^2|X \leqslant \eta\}.$$

Analogously, noting that

$$\zeta - \mu_X = \int_{-\infty}^{\infty}(\zeta - \xi)P_X(\mathrm{d}\xi),$$

we obtain

$$\int_\eta^\infty [F_X^{(2)}(\zeta) - (\zeta - \mu_X)]\ \mathrm{d}\zeta$$
$$= \int_\eta^\infty\left[\int_\zeta^\infty(\xi-\zeta)P_X(\mathrm{d}\xi)\right]\mathrm{d}\zeta$$
$$= \int\int_{\substack{\eta\leqslant\zeta\\ \zeta\leqslant\xi}}(\xi - \zeta)\ P_X(\mathrm{d}\xi)\ \mathrm{d}\zeta$$
$$= \int_\eta^\infty\left[\int_\eta^\xi(\xi - \zeta)\ \mathrm{d}\zeta\right] P_X(\mathrm{d}\xi)$$
$$= \frac{1}{2}\int_\eta^\infty(\xi - \eta)^2\ P_X(\mathrm{d}\xi)$$
$$= \frac{1}{2}P\{X \geqslant \eta\}E\{(X - \eta)^2|X \geqslant \eta\}.$$

References

Bawa, V.S., 1982. Stochastic dominance: A research bibliography. Management Science 28, 698–712.

Fishburn, P.C., 1964. Decision and Value Theory, Wiley, New York.

Fishburn, P.C., 1977. Mean-risk analysis with risk associated with below target returns. The American Economic Review 67, 116–126.

Fishburn, P.C., 1980. Stochastic dominance and moments of distributions. Mathematics of Operations Research 5, 94–100.

Hanoch, G., Levy, H., 1969. The efficiency analysis of choices involving risk. Review of Economic Studies 36, 335–346.

Hardy, G.H., Littlewood, J.E., Pólya, G., 1934. Inequalities, Cambridge University Press, Cambridge, MA.

Kendall, M.G., Stuart, A., 1958. The Advanced Theory of Statistics, vol. 1: Distribution Theory, Griffin, London.

Konno, H., Yamazaki, H., 1991. Mean-absolute deviation portfolio optimization model and its application to Tokyo stock market. Management Science 37, 519–531.

Levy, H., 1992. Stochastic dominance and expected utility: Survey and analysis. Management Science 38, 555–593.

Markowitz, H.M., 1952. Portfolio selection. Journal of Finance 7, 77–91.

Markowitz, H.M., 1959. Portfolio Selection. Wiley, New York.

Markowitz, H.M., 1987. Mean–Variance Analysis in Portfolio Choice and Capital Markets. Blackwell, Oxford.

Markowitz, H.M., Todd, P., Xu, G., Yamane, Y., 1993. Computation of mean–semivariance efficient sets by the critical line algorithm. Annals of Operations Research 45, 307–317.

Porter, R.B., 1974. Semivariance and stochastic dominance: A comparison. The American Economic Review 64, 200–204.
Porter, R.B., Gaumnitz, J.E., 1972. Stochastic dominance versus mean–variance portfolio analysis. The American Economic Review 62, 438–446.
Pratt, J.W., Raiffa, H., Schlaifer, R., 1995. Introduction to Statistical Decision Theory, MIT Press, Cambridge, MA.
Rothschild, M., Stiglitz, J.E., 1970. Increasing risk: I. A definition. Journal of Economic Theory 2, 225–243.
Whitmore, G.A, Findlay, M.C. (Eds.), 1978. Stochastic Dominance: An Approach to Decision-Making Under Risk, Heath, Lexington, MA.
Yitzhaki, S., 1982. Stochastic dominance, mean variance, and Gini's mean difference. The American Economic Review 72, 178–185.
Zenios, S.A., Kang, P., 1993. Mean-absolute deviation portfolio optimization for mortgage-backed securities. Annals of Operations Research 45, 433–450.