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Multiple Criteria Optimization and Decisions under Risk[1]

by

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Abstract: The mathematical background of multiple criteria optimization (MCO) is closely related to the theory of decisions under uncertainty. Most of the classical solution concepts commonly used in the MCO methodology have their origins in some approaches to handle uncertainty in the decision analysis. Nevertheless, the MCO as a separate discipline has developed several advanced tools of the interactive analysis leading to effective decision support techniques with successful applications. Progress made in the MCO tools raises a question of possible feedback to the decision making under risk. The paper shows how the decisions under risk, and specifically the risk aversion preferences, can be modeled within the MCO methodology. This provides a methodological basis allowing to take advantages of the interactive multiple criteria techniques for decision support under risk.

Keywords: Multiple Criteria Optimization, Decisions under Uncertainty, Decisions under Risk, Risk Aversion

1. Introduction

The multiple criteria optimization (MCO) is commonly considered a relatively young methodology. However, the theoretical basis of MCO is closely related to the decision theory with its very early roots in the eighteen century (Stadler, 1979). Later, the theory of decisions under uncertainty had a crucial impact on the initial MCO concepts. Especially, the utility concepts influenced the classical MCO developments (Fishburn, 1964; Keeney and Raiffa, 1976). Actually, most of the classical solution concepts commonly used in MCO have their roots (or equivalents) in some approaches to handle uncertainty in the decision analysis.

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The MCO as a separate discipline has experienced enormous development during recent years (Steuer et al., 1996). Following Charnes and Cooper (1961) efforts to operationalize the MCO approaches, a gamut of techniques and algorithms arrived. Several advanced tools of the interactive analysis have been introduced to define a decision support process. They depend on additional preference information gained interactively from the DM, allowing simultaneously the DM to learn the problem during the process with possible evolvement of the preferences. In particular the reference point approach (Wierzbicki, 1982) has led to effective decision support techniques with successful applications (Wierzbicki et al., 2000).

The methods of decisions under risk were operationalized by Markowitz (1952) with his seminal mean-variance model. Since then many authors have pointed out that the mean-variance model is, in general, not consistent with stochastic dominance rules or other axiomatic models of decisions under risk. Nevertheless, the decisions under risk and particularly portfolio optimization remain mostly within the mean-variance methodology. Progress made in the MCO tools raises a question of possible feedback to the decision making under risk. The paper shows how the decisions under risk, and specifically the risk aversion preferences, can be modeled within the MCO methodology. It systematizes commonly known relations as well as introduces new models. This provides a methodological basis allowing to take advantages of the interactive multiple criteria techniques for decision support under risk.

The paper is organized as follows. In the next section, we consider decision problems under uncertainty where the decision is based on maximization of a scalar outcome with various realizations under several scenarios. We show that the rational preference model leads then to the Pareto efficiency with respect to the realizations under scenarios understood as multiple criteria. We demonstrate also how the classical solution concepts commonly used in the multiple criteria optimization can be originated from approaches to handle uncertainty in the decision analysis. Further, in Section 3 we deal with decisions under risk. We show that the case of equally probable scenarios leads to the concept of symmetric optimization (efficiency) of multiple criteria corresponding to realizations under scenarios. The concept is further extended to the distribution approach equivalent to the rules of the first degree stochastic dominance (FSD). Two alternative mixed integer linear programming criteria modifications are introduced, thus allowing us to represent the decisions under risk within the standard MCO methodology. In Section 4 we focus on risk averse preferences. The case of equally probable scenarios leads us then to the concept of equitable optimization (efficiency) of multiple criteria corresponding to realizations under scenarios. Extension to the distribution approach is equivalent to the rules of the second degree stochastic dominance (SSD). Two alternative linear programming criteria modifications are introduced allowing us to represent the risk averse preferences with the standard MCO methodology. This results, in particular, in multiple criteria linear programming models for portfolio optimization.

2. Decisions under uncertainty and Pareto efficiency

2.1. Pareto efficiency

Let us consider a decision problem under uncertainty where the decision is based on the maximization of a scalar (real valued) outcome. The final outcome is uncertain and only its realizations under various scenarios are known. Exactly, for each scenario S_i $(i = 1, \dots, m)$ the corresponding outcome realization is given as a function of the decision variables $y_i = f_i(\mathbf{x})$. We are interested in larger outcomes under each scenario. Hence, the decision under uncertainty can be considered a multiple criteria optimization problem:

$$\max \{\mathbf{f}(\mathbf{x}) \;:\; \mathbf{x} \in Q\}, \tag{1}$$

where :

$\mathbf{f} = (f_1, \dots, f_m)$	is a vector–function that maps the decision space $X = R^n$ into the criterion space $Y = R^m$,
$Q \subset X$	denotes the feasible set,
$\mathbf{x} \in X$	denotes the vector of decision variables.

The elements of the criterion space we refer to as achievement vectors. An achievement vector $\mathbf{y} \in Y$ is attainable if it expresses outcomes of a feasible solution $\mathbf{x} \in Q$ $(\mathbf{y} = \mathbf{f}(\mathbf{x}))$.

From the perspective of decisions under uncertainty, model (1) only specifies that we are interested in maximization of all objective functions f_i for $i \in I = \{1, 2, \dots, m\}$. In order to make it operational, one needs to assume some solution concept specifying what it means to maximize multiple objective functions. The solution concepts are defined by properties of the corresponding preference model. We assume that solution concepts depend only on evaluation of the achievement vectors (outcomes) while not taking into account any other solution properties not represented within the achievement vectors. Thus, we can limit our considerations to the preference model in the space of achievement vectors. The preference model is completely characterized by the relation of weak preference (Vincke, 1992), denoted hereafter with $\succeq$. Namely, the corresponding relations of strict preference $\succ$ and indifference $\cong$ are defined by the following formulas:

$$\mathbf{y}' \succ \mathbf{y}'' \quad \Leftrightarrow \quad (\mathbf{y}' \succeq \mathbf{y}'' \quad \text{and} \quad \mathbf{y}'' \not\succeq \mathbf{y}'),$$
$$\mathbf{y}' \cong \mathbf{y}'' \quad \Leftrightarrow \quad (\mathbf{y}' \succeq \mathbf{y}'' \quad \text{and} \quad \mathbf{y}'' \succeq \mathbf{y}').$$

The standard preference model related to the Pareto efficient solution concept assumes that the preference relation $\succeq$ is *reflexive*:

$$\mathbf{y} \succeq \mathbf{y}, \tag{2}$$

transitive:

$$(\mathbf{y}' \succeq \mathbf{y}'' \quad \text{and} \quad \mathbf{y}'' \succeq \mathbf{y}''') \quad \Rightarrow \quad \mathbf{y}' \succeq \mathbf{y}''', \tag{3}$$

and *strictly monotonic*:

$$\mathbf{y} + \varepsilon \mathbf{e}_i \succ \mathbf{y} \quad \text{for } \varepsilon > 0; \ i = 1, \dots, m, \tag{4}$$

where $\mathbf{e}_i$ denotes the i–th unit vector in the criterion space. The last assumption expresses that for each individual objective function more is better (maximization). The preference relations satisfying axioms (2)–(4) are called hereafter *rational preference relations*. The rational preference relations allow us to formalize the Pareto efficiency concept with the following definitions. We say that achievement vector $\mathbf{y}'$ rationally dominates $\mathbf{y}''$ ($\mathbf{y}' \succ_r \mathbf{y}''$), iff $\mathbf{y}' \succ \mathbf{y}''$ for all rational preference relations $\succeq$. We say that feasible solution $\mathbf{x} \in Q$ is a *Pareto efficient* solution of the multiple criteria problem (1), iff $\mathbf{y} = \mathbf{f}(\mathbf{x})$ is rationally nondominated.

The relation of weak rational dominance $\mathbf{y}' \succeq_r \mathbf{y}''$ may be expressed in terms of the vector inequality $\mathbf{y}' \geqq \mathbf{y}''$. Hence, we can state that a feasible solution $\mathbf{x}^0 \in Q$ is a Pareto efficient solution of the multiple criteria problem (1), if and only if, there does not exist $\mathbf{x} \in Q$ such that $\mathbf{f}(\mathbf{x}) \geq \mathbf{f}(\mathbf{x}^0)$. The latter refers to the commonly used definition of the (Pareto) efficient solutions as feasible solutions for which one cannot improve any criterion without worsening another (Steuer, 1986). In other words, decision problem under uncertainty defined by a finite set scenarios S_i ($i = 1, \dots, m$) and the corresponding outcome realizations $y_i = f_i(\mathbf{x})$ may be considered a standard multiple criteria optimization (1). However, the axiomatic definition of the rational preference relation will allow us to introduce additional properties of the preferences related to the principles of choice under risk.

2.2. Multiple criteria optimization and decision support

There usually does not exist an outcome vector that dominates all others with respect to all the criteria. Thus in terms of strict mathematical relations we cannot distinguish the best outcome vector. All the Pareto efficient solutions are incomparable on the basis of the specified set of criteria. In theory, one may consider a multiple criteria optimization as a problem depending on identification of the entire set of efficient solutions. We are interested, however, in an operational use of multiple criteria analysis to help the decision maker (DM) to select one efficient solution for implementation.

Efficient solutions of the multiple criteria problem (1) can be generated with simple scalarizations of the problem. Most of them are based on the maxisum approach:

$$\max \{ \sum_{i=1}^{m} f_i(\mathbf{x}) \ : \quad \mathbf{x} \in Q \}, \tag{5}$$

or on the maximin approach:

$$\max \{ \min_{i=1,\dots,m} f_i(\mathbf{x}) \ : \quad \mathbf{x} \in Q \}. \tag{6}$$

However, the latter generates an efficient solution only in the case of a unique optimal solution. In the general case, the optimal set of (6) contains an efficient solution but it may also include some dominated ones. Therefore, some additional refinement (regularization) is necessary to select the optimal solution which is efficient and the maximin scalarization is regularized by the additional maxisum term thus generating the augmented maximin problem (Steuer, 1986):

$$\max \{ \min_{i=1,\ldots,m} f_i(\mathbf{x}) + \varepsilon \sum_{i=1}^{m} f_i(\mathbf{x}) \ : \quad \mathbf{x} \in Q \}, \tag{7}$$

where ε is an arbitrarily small positive parameter. In terms of decisions under uncertainty, the maximin approach represents a pessimistic solution concept of the worst scenario achievement maximization. One may also consider the maximax approach to represent an optimistic solution concept of the best scenario maximization. Similar to the maximin approach, it needs a regularization for the case of nonunique optimal solution thus resulting in the following formulation:

$$\max \{ \max_{i=1,\ldots,m} f_i(\mathbf{x}) + \varepsilon \sum_{i=1}^{m} f_i(\mathbf{x}) \ : \quad \mathbf{x} \in Q \}. \tag{8}$$

The maxisum approach (5) represents the so-called Laplace criterion of selection under uncertainty. Namely, maximization of the total outcome is equivalent to the maximization of the average outcome (arithmetic mean). The latter represents the expected value maximization under assumption that all scenarios are equally probable which is just the concept of the Laplace criterion. One may consider the weighting approach:

$$\max \{ \sum_{i=1}^{m} w_i f_i(\mathbf{x}) \ : \quad \mathbf{x} \in Q \}, \tag{9}$$

with positive weights w_i summing to 1. The approach generates Pareto efficient solution and it can be interpreted as the expected value maximization with various subjective probabilities (or importance) of several scenarios. Unfortunately, the weighting approach does not provide us with a complete parameterization of the entire Pareto efficient set, thus restricting the preference model. Actually, in the case of a discrete (or nonconvex) feasible set Q, there exist Pareto efficient solutions that cannot be identified as optimal solutions to problem (9) with any set of positive weights (Ogryczak, 1997). This flaw of the weighting approach (9) can be overcome with the weights considered in the maximin aggregation (7) as used within the reference point approaches (Wierzbicki, 1982) discussed below.

In general, it is very difficult to identify and formalize the DM preferences at the beginning of the decision process. Therefore, a decision support process is needed which depends on additional preference information gained interactively from the DM, allowing simultaneously the DM to learn the problem during the process with possibly evolving preferences. This can be effectively achieved with the so-called quasi-satisficing approach to multiple criteria decision problems (Wierzbicki et al., 2000). The best formalization of the quasi-satisficing approach to multiple criteria optimization was proposed and developed mainly by Wierzbicki (1982) as the reference point method. The reference point method (RPM) is an interactive technique where the DM specifies preferences in terms of aspiration levels (reference point), i.e., by introducing desired (acceptable) levels for several criteria. Depending on the specified aspiration levels, a special scalarizing achievement function is built which, when optimized, generates an efficient solution to the problem. The scalarizing achievement function may be directly interpreted as expressing utility to be maximized. However, to keep the discussion consistent we will assume that the scalarizing achievement function is minimized (thus representing dis-utility). The computed efficient solution is presented to the DM as the current solution in a form that allows comparison with the previous ones and modification of the aspiration levels if necessary.

While building the scalarizing achievement function the following properties of the preference model are assumed. First of all, for any individual outcome y_i more is preferred to less (maximization). To meet this requirement the function must be strictly decreasing with respect to each outcome. Second, a solution with all individual outcomes y_i equal to the corresponding aspiration levels is preferred to any solution with at least one individual outcome worse (smaller) than its aspiration level. Thus, similar to the goal programming approaches (Charnes and Cooper, 1961), the aspiration levels are treated as the targets but following the quasi-satisficing approach they are interpreted consistently with basic concepts of efficiency in the sense that the optimization is continued even when the target point has been reached already.

The generic scalarizing achievement function takes then the following form (Wierzbicki, 1982):

$$\max_{1\leq i\leq m}\{s_i(a_i,f_i(\mathbf{x}))\}+\varepsilon\sum_{i=1}^{m}s_i(a_i,f_i(\mathbf{x})), \tag{10}$$

where ε is an arbitrary small positive number and $s_i : R^2 \to R$, for $i = 1,\ldots,m$, are the individual achievement functions measuring actual achievement of the i-th outcome with respect to the corresponding aspiration levels a_i. For any reference value a_i, function $s_i(a_i, y_i)$ must be strictly decreasing with respect to y_i (the i-th outcome) and it has to take value 0 for $y_i = a_i$.

Various functions s_i provide a wide modeling environment for measuring individual achievements (Wierzbicki et al., 2000; Ogryczak, 1997). For the sake of computational robustness, the piece wise linear functions s_i are usually employed. In the simplest models, they take a form of two segment piece wise linear functions:

$$s_i(a_i, y_i) = \begin{cases} v_i^n(a_i - y_i), & \text{for} \quad y_i \leq a_i \\ v_i^p(a_i - y_i), & \text{for} \quad y_i > a_i \end{cases} \tag{11}$$

where v_i^n and v_i^p are positive weights corresponding to underachievements and overachievements, respectively, for the i-th outcome (v_i^n is usually much larger than v_i^p).

Let us consider a specific case when the best possible values of the individual outcomes (the so-called utopia or ideal point) are used as the aspiration levels, i.e., $a_i = y_i^* = \max\,\{f_i(\mathbf{x}) : \ \mathbf{x} \in Q\}$. The achievement function (11) can be simplified then to $v_i^n(y_i^* - y_i)$ as in goal programming (GP) models (Ogryczak, 2001). Note that in decisions under uncertainty, following Savage (1954), the minimization of $\max_{i=1,\dots,m}\,(y_i^* - y_i)$ is used as the so-called minimax regret criterion. This commonly accepted decision selection rule expresses the minimization of the (maximum) regret due to the opportunity loss. Namely, after decisions have been made and the scenarios have occurred, the DMs may experience regret because they now know what scenario have taken place and may wish that they had selected different action. The regret is the difference between the outcome that could be achieved with perfect knowledge of the future and the outcome that was achieved from the selected solution. Hence, the scalarizing achievement function (10)–(11) is a generalization of the minimax regret criterion used in the decision analysis. The RPM, similar to the interactive multiple goal programming (IMGP) techniques, extends the maximum regret criterion with a capability to consider various reference achievements to define opportunity loss (regret) as well as it allows us to scale differently several regret measures. Moreover, in the RPM (but not in IMGP), the regularization is introduced to guarantee Pareto efficiency of all the generated solutions.

The reference point method was later extended to permit additional information from the DM. It is implemented in the form of so-called aspiration/reservation based decision support (ARBDS) which in addition to the main target (aspiration) levels a_i employs also reservation levels r_i, so that the DM can specify desired as well as required values for given outcomes. This allows an implicit definition of weights leaving aspiration and reservation levels as the exclusive control parameters. The ARBDS techniques were implemented in several decision systems with many successful applications (Lewandowski and Wierzbicki, 1989; Wierzbicki et al., 2000). Since the problems of decision under uncertainty can be treated as standard MCO problems (1), the ARBDS methodology provides us with effective tools to solve interactively decision problems under uncertainty. It is important that various Pareto efficient solutions can be found in this way (compare Kaliszewski (1994) with respect to limitations caused by the positive value of ε) and, therefore, the methodology is capable to meet various rational preference model connected to decisions under uncertainty. Moreover, it can be applied to continuous as well as discrete feasible sets. In further sections we will show that the ARBDS approaches can be also applied to decisions under risk.

3. Decisions under risk and symmetric efficiency

3.1. Symmetric efficiency

As discussed in the previous section, any decision problem under uncertainty defined by a finite set scenarios S_i $(i = 1, \ldots, m)$ and the corresponding outcome realizations $y_i = f_i(\mathbf{x})$ may be considered a standard multiple criteria optimization (1). In a decision problem under risk the probabilities of several scenarios are also known. The decision outcome is then a random variable and it is described by a distribution of values y_i with probabilities p_i. Although the decision problem can still be expressed as the multiple criteria optimization (1), the corresponding criteria $y_i = f_i(\mathbf{x})$ as attributed with probabilities cannot be treated as completely independent criteria. Achievement vector $\mathbf{y}$ represents now a lottery (the values of several tickets in a lottery). Hence, while representing the decisions under risk the problem (1) is no longer a standard multiple criteria optimization with a possible selection of any rational preference relation.

Let us focus on the case of equally probable scenarios. Assuming that all the scenario probabilities p_i are given as rational numbers, the decision problem can easily be transformed into an equivalent problem with equally probable scenarios. Note that the transformation itself is conceptually very simple as requiring only multiple replications of some scenarios. Nevertheless, it may cause exploding problem size with respect to the resulting number of scenarios. Therefore, we assume the equally probable scenarios to introduce some formal concepts while later this assumption will be relaxed for computational approaches.

Under the assumption of equally probable scenarios, the achievement vector $\mathbf{y} = (y_1, y_2, \ldots, y_m)$ within the multiple criteria problem (1) represents an uncertain outcome as a lottery with m equally probable $(p_i = p = \frac{1}{m})$ tickets providing gains y_i $(i = 1, \ldots, m)$, respectively. In order to interpret the achievement vectors as lotteries one needs to focus on the distribution of outcome values while ignoring their ordering. That means, in the multiple criteria optimization problem (1) we are interested in a set of values of the criteria without taking into account which criterion is taking a specific value. Such a requirement is mathematically formalized as the property of impartiality of the preference relation. We say that a preference relation $\succeq$ is *impartial* (anonymous, symmetric) if

$$(y_{\tau(1)}, y_{\tau(2)}, \ldots, y_{\tau(m)}) \cong (y_1, y_2, \ldots, y_m) \tag{12}$$

for any permutation τ of I.

The rational preference relations satisfying the requirement of impartiality we call hereafter *impartial rational preference relations*. The impartial rational preference relations allow us to introduce the concept of symmetric efficiency by the following definitions. We say that achievement vector $\mathbf{y}'$ *symmetrically dominates* $\mathbf{y}''$ ($\mathbf{y}' \succ_s \mathbf{y}''$), iff $\mathbf{y}' \succ \mathbf{y}''$ for all impartial rational preference relations $\succeq$. We say that feasible solution $\mathbf{x} \in Q$ is a *symmetrically efficient* solution of the multiple criteria problem (1), iff $\mathbf{y} = \mathbf{f}(\mathbf{x})$ is symmetrically nondominated.

Note that an impartial rational preference relation must satisfy axioms (2)–(4) and the requirement (12). Hence, achievement vector $\mathbf{y}' \in Y$ symmetrically dominates $\mathbf{y}'' \in Y$, or $\mathbf{y}''$ is symmetrically dominated by $\mathbf{y}'$, if there exist permutations τ' and τ'' such that $y'_{\tau'(i)} \geq y''_{\tau''(i)}$ for all $i \in I$ and for at least one index i_0 strict inequality holds (i.e., $y'_{\tau'(i_0)} > y''_{\tau''(i_0)}$). The symmetric dominance relation may be illustrated with the so–called domination structure (Nakayama et al., 1985), i.e. a point–to–set map

$$D(\mathbf{y}) = \{\mathbf{d} \in Y \ : \quad \mathbf{y} \succ \mathbf{y} + \mathbf{d}\} \cup \{\mathbf{0}\}. \tag{13}$$

For the standard rational dominance relation, the sets $D(\mathbf{y})$ are independent of $\mathbf{y}$ and they take the form of the nonnegative orthant. The domination structure of the symmetric dominance depends on the location of an achievement vector $\mathbf{y}$ relative to the absolute equity line ($y_1 = y_2 = \cdots = y_m$). In the general case, the set $D(\mathbf{y})$ is not a cone and it is not convex. Figure 1 shows $D(\mathbf{y})$ fixed at $\mathbf{y}$, i.e. the set $\mathbf{y} + D(\mathbf{y})$.

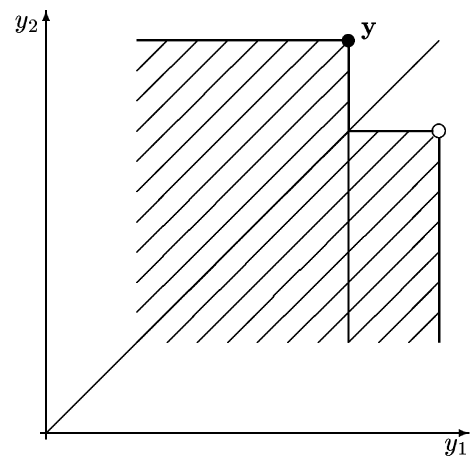

Figure 1. Symmetric dominance structure in R^2

The relation of symmetric domination can be expressed as domination of the achievement vectors with coefficients ordered in nonincreasing order. This can be mathematically formalized with the ordering map $\Theta : R^m \to R^m$ such that $\Theta(\mathbf{y}) = (\theta_1(\mathbf{y}), \theta_2(\mathbf{y}), \ldots, \theta_m(\mathbf{y}))$, where $\theta_1(\mathbf{y}) \leq \theta_2(\mathbf{y}) \leq \cdots \leq \theta_m(\mathbf{y})$ and there exists a permutation τ of set I such that $\theta_i(\mathbf{y}) = y_{\tau(i)}$ for $i = 1, \ldots, m$. The following proposition is valid (Podinovskii, 1975).

Proposition 3.1 *An achievement vector $\mathbf{y}' \in Y$ symmetrically dominates $\mathbf{y}'' \in Y$, if and only if $\Theta(\mathbf{y}')$ dominates $\Theta(\mathbf{y}'')$, i.e. $\theta_i(\mathbf{y}') \geq \theta_i(\mathbf{y}'')$ for all $i \in I$ and for at least one index i_0 strict inequality holds (i.e., $\theta_{i_0}(\mathbf{y}') > \theta_{i_0}(\mathbf{y}'')$).*

Proposition 3.1 permits one to express symmetric efficiency for problem (1) in terms of the standard efficiency for the multiple criteria problem with objectives $\Theta(\mathbf{f}(\mathbf{x}))$:

$$\max \{(\theta_1(\mathbf{f}(\mathbf{x})), \theta_2(\mathbf{f}(\mathbf{x})), \ldots, \theta_m(\mathbf{f}(\mathbf{x}))) \; : \; \mathbf{x} \in Q\}. \tag{14}$$

A feasible solution $\mathbf{x} \in Q$ is a symmetrically efficient solution of the multiple criteria problem (1), if and only if it is an efficient solution of the multiple criteria problem (14).

Note that the maximin (7) and the maximax (8) approaches to the original multiple criteria problem (1) can also be expressed in terms of the ordered outcomes of problem (14). Hence, these approaches generate symmetric efficient solutions. The same applies to the maxisum (5) approach but it cannot be extended to the weighting approaches (9). One may apply the weighting approach to the ordered problem (14) but this will result in the ordered weighted averaging aggregation (Yager, 1988) applied to the original problem (1).

The quantity $\theta_1(\mathbf{y})$ representing the worst outcome can be easily computed directly by the LP maximization:

$$\theta_1(\mathbf{y}) = \max \; t_1 \quad \text{s.t.} \quad t_1 \leq y_i \quad \text{for } i = 1, \ldots, m.$$

Similar formula can be given for any $\theta_k(\mathbf{y})$ although requiring the use of integer variables. Namely, for any $k = 1, 2, \ldots, m$ the following formula is valid:

$$\begin{aligned} \theta_k(\mathbf{y}) \quad = \quad & \max t_k \\ \text{s.t.} \quad & t_k - y_i \leq M z_{ki}, \; z_{ki} \in \{0,1\} \quad \text{for } i = 1, \ldots, m, \\ & \sum_{i=1}^{m} z_{ki} \leq k - 1. \end{aligned} \tag{15}$$

where M is a sufficiently large constant (larger than any possible difference between various individual outcomes y_i). Note that for $k = 1$ all the binary variables z_{1i} are enforced to 0 thus reducing the optimization to the standard LP model for that case.

The entire ordered multiple criteria model (14) can be formulated as the following mixed integer multiple criteria problem:

$$\begin{aligned} \max \quad & [t_1, t_2, \ldots, t_m] \\ \text{s.t.} \quad & \\ & t_k - f_i(\mathbf{x}) \leq M z_{ki} \qquad \text{for } i = 1, \ldots, m; \; k = 1, \ldots, m, \\ & z_{ki} \in \{0,1\} \qquad \text{for } i = 1, \ldots, m; \; k = 1, \ldots, m, \\ & \sum_{i=1}^{m} z_{ki} \leq k - 1 \qquad \text{for } k = 1, \ldots, m, \\ & \mathbf{x} \in Q. \end{aligned} \tag{16}$$

Mixed integer formulation of the multiple criteria problem (14) implies that its efficient set may be not connected. Indeed the symmetric efficient set may be in general not connected as illustrated in the following example. Therefore, integer variables are necessary to formulate its ordered equivalent.

EXAMPLE 3.1 *Let us consider a bicriteria problem:*

$$\max\{(x_1, x_2): \quad x_1 + 3x_2 \leq 63, \quad 5x_1 + 4x_2 \leq 117, \quad x_1 \geq 0, \quad x_2 \geq 0\}.$$

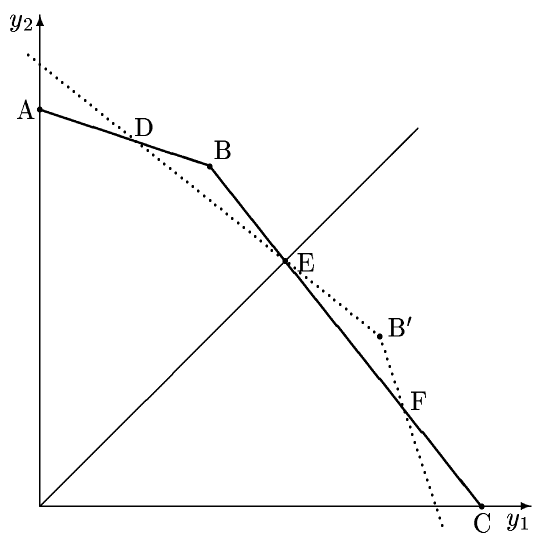

Figure 2. Not connected symmetric efficient set $\overline{\mathrm{DB}} \cup \overline{\mathrm{BE}} \cup \overline{\mathrm{FC}}$

Figure 2 gives the graphical illustration of the problem in the objective space (y_1, y_2)*. However, due to the identity objective functions* $y_1 = x_1$ *and* $y_2 = x_2$*, it can be directly interpreted in the decision space* (x_1, x_2)*. The standard efficient set consists of two line segments* $\overline{\mathrm{AB}}$ *and* $\overline{\mathrm{BC}}$*, where* A=(0,21)*,* B=(9,18) *and* C=(23.4,0)*. While taking into account symmetric dominance we eliminate segments* $\overline{\mathrm{AD}}$ *and* $\overline{\mathrm{EF}}$*. Thus, the symmetric efficient set consists of three segments:* $\overline{\mathrm{DB}}$*,* $\overline{\mathrm{BE}}$ *and* $\overline{\mathrm{FC}}$*. Note that the symmetric efficient set is not connected.*

3.2. Distribution approach

The ordered achievement vectors describe a distribution of outcomes generated by a given decision **x**. In the case when there exists a finite set of all possible outcomes of the individual objective functions, we can directly deal with the distribution of outcomes described by frequencies of several outcomes. Let $V = \{v_1, v_2, \ldots, v_r\}$ (where $v_1 < v_2 < \cdots < v_r$) denote the set of all attainable outcomes (all possible values of the individual objective functions f_i for $\mathbf{x} \in Q$). We introduce integer functions $h_k(\mathbf{y})$ $(k = 1, \ldots, r)$ expressing the number of values v_k taken in the achievement vector $\mathbf{y}$. Having defined the functions h_k we can introduce cumulative distribution functions:

$$\bar{h}_k(\mathbf{y}) = \sum_{l=1}^{k} h_l(\mathbf{y}) \ , \quad \text{for } k = 1, \ldots, r. \tag{17}$$

The function $\bar{h}_k$ expresses the number of outcomes smaller or equal to v_k. Since we want to maximize all the outcomes, we are interested in the minimization of all the functions $\bar{h}_k$. The following assertion is valid (Ogryczak, 1997). For achievement vectors $\mathbf{y}', \mathbf{y}'' \in V^m$,

$$\Theta(\mathbf{y}') \geq \Theta(\mathbf{y}'') \quad \Leftrightarrow \quad \bar{\mathbf{h}}(\mathbf{y}') \leq \bar{\mathbf{h}}(\mathbf{y}''). \tag{18}$$

Equivalence (18) permits one to express symmetric efficiency for problem (1) in terms of the standard efficiency for the multiple criteria problem with objectives $\bar{\mathbf{h}}(\mathbf{f}(\mathbf{x}))$:

$$\min \ \{(\bar{h}_1(\mathbf{x}), \bar{h}_2(\mathbf{x}), \ldots, \bar{h}_r(\mathbf{x})) \ : \quad \mathbf{x} \in Q\}. \tag{19}$$

PROPOSITION 3.2 *A feasible solution $\mathbf{x} \in Q$ is a symmetrically efficient solution of the multiple criteria problem (1), if and only if it is an efficient solution of the multiple criteria problem (19).*

The quantity $\bar{h}_k(\mathbf{y})$ can be computed directly by the minimization:

$$\begin{aligned} \bar{h}_k(\mathbf{y}) \quad &= \quad \min \sum_{i=1}^{m} z_{ki} \\ &\text{s.t.} \quad v_{k+1} - y_i \leq M z_{ki}, \ z_{ki} \in \{0, 1\} \quad \text{for } i = 1, \ldots, m, \end{aligned}$$

where M is a sufficiently large constant. Hence, the multiple criteria model (19) can be formulated as the following mixed integer multiple criteria problem:

$$\begin{aligned} \min \quad & \left[\sum_{i=1}^{m} z_{1i}, \ \sum_{i=1}^{m} z_{2i}, \ldots, \ \sum_{i=1}^{m} z_{r-1,i} \right] \\ \text{s.t.} \quad & \\ & v_{k+1} - f_i(\mathbf{x}) \leq M z_{ki} \quad \text{for } i = 1, \ldots, m, k = 1, \ldots, r-1, \\ & z_{ki} \in \{0, 1\} \quad \text{for } i = 1, \ldots, m, k = 1, \ldots, r-1, \\ & \mathbf{x} \in Q. \end{aligned} \tag{20}$$

Note that $\bar{h}_r(\mathbf{y}) = m$ for any $\mathbf{y}$ which means that the r-th criterion is always constant and therefore redundant in (19). Moreover

$$v_r + \sum_{k=1}^{r-1} \frac{v_k - v_{k+1}}{m} \bar{h}_k(\mathbf{y}) = \frac{1}{m} \sum_{k=1}^{r} v_k h_k(\mathbf{y}) = \frac{1}{m} \sum_{i=1}^{m} y_i = E[Y],$$

where the weights $(v_k - v_{k+1})/m$ are negative. Hence, when applying to problem (19) the weighting approach based on the maximization (note the sign change) of the weighted combination with weights $w_k = (v_k - v_{k+1})/m$ for $k = 1, \ldots, r-1$, one gets a symmetric efficient solution equivalent to the maximization of the mean outcome (the expected value of the corresponding random variable). In other words, the maximization of the mean (expected) outcome is equivalent to a specific weighting aggregation (9) of the multiple criteria optimization problem (19). The same can be shown for the expected utility maximization (Levy, 1992). One may notice that for any utility function u

$$E[u(Y)] = \frac{1}{m} \sum_{i=1}^{m} u(y_i) = u(v_r) + \sum_{k=1}^{r-1} \frac{u(v_k) - u(v_{k+1})}{m} \bar{h}_k(\mathbf{y}). \tag{21}$$

Certainly, in the case of a strictly increasing utility function u, the corresponding weight coefficients are negative. Therefore, maximization of the expected utility generates an efficient solution of problem (19) and consequently a symmetric efficient solution to the original multiple criteria problem (1). On the other hand, as based on the weighting approach (9), the expected utility approach does not allows us to identify all the symmetric efficient solutions. Especially, that problem (20) is discrete even for a convex feasible set Q.

3.3. FSD and general MCO models

Vector $\bar{\mathbf{h}}(\mathbf{y})$ has a lucid interpretation when its coefficients are considered as a function of the corresponding values v_k, i.e., the pairs $(v_k, \bar{h}_k(\mathbf{y})/m)$ for $k = 1, \ldots, r$ are considered. The function can be extended to a right continuous non decreasing function of outcome value v. The extension is based on the definition of $\bar{h}_v(\mathbf{y})$ for any $v \in R$ as a number of outcomes y_i less or equal to v, i.e.

$$\bar{h}_v(\mathbf{y}) = \begin{cases} m, & \text{for} \quad v \geq v_r \\ \bar{h}_k(\mathbf{y}), & \text{for} \quad v_k \leq v < v_{k+1} \\ 0, & \text{for} \quad v < v_1 \end{cases}$$

Note that $\frac{1}{m}\bar{h}_v(\mathbf{y})$ considered as a function of v represents the (right continuous) cumulative distribution function of random variable Y generating the outcomes distribution y_i:

$$\frac{1}{m}\bar{h}_v(\mathbf{y}) = P\{Y \leq v\} = F_Y(v). \tag{22}$$

Hence, inequality $\bar{\mathbf{h}}(\mathbf{y}') \leq \bar{\mathbf{h}}(\mathbf{y}'')$ represents the relation of the *first order stochastic dominance (FSD)* (Levy, 1992). Recall that the weak FSD dominance is defined as

$$Y' \succeq_{FSD} Y'' \quad \Leftrightarrow \quad F_{Y'}(v) \leq F_{Y''}(v) \quad \forall v \in R, \tag{23}$$

while the strict FSD dominance is defined in the standard way

$$Y' \succ_{FSD} Y'' \quad \Leftrightarrow \quad Y' \succeq_{FSD} Y'' \quad \text{and} \quad Y'' \not\succeq_{FSD} Y'.$$

We say that random variable Y' dominates Y'' according to FSD if $F_{Y'}(v) \leq F_{Y''}(v)$ for any $v \in R$, and there exists $v_0 \in R$ such that a strict inequality holds.

The relation of first stochastic dominance is crucial while comparing risky profit defined by random variables (Whitmore and Findlay, 1978). In a general case it is based on the comparison of the infinite number (continuum) of criteria defined as values of the cumulative distribution functions $F(v)$ for every possible real target $v \in R$. Note that having a cumulative distribution function representing a lottery $\mathbf{y}$, one can define the order outcomes $\theta_i(\mathbf{y})$ as values of the corresponding i/m-quantiles. For this purpose one may consider the left continuous generalized inverse of the cumulative distribution function:

$$F_Y^{(-1)}(p) = \inf \{v \ : \quad F_Y(v) \geq p\} \quad \text{for } 0 < p \leq 1. \tag{24}$$

Values $F_Y^{(-1)}(\beta)$ represent β-quantiles of Y providing alternative characterization of FSD as:

$$Y' \succeq_{FSD} Y'' \quad \Leftrightarrow \quad F_{Y'}^{(-1)}(\beta) \geq F_{Y''}^{(-1)}(\beta) \quad \forall \beta \in (0,1]. \tag{25}$$

This is again a comparison model based on the infinite number (continuum) of criteria defined as values of the quantile functions $F^{(-1)}(\beta)$ for every possible tolerance level $0 < \beta \leq 1$. If random variable Y represents a lottery $\mathbf{y}$ with m equally probable tickets, then

$$\theta_i(\mathbf{y}) = F_Y^{(-1)}(\frac{i}{m}) \quad \text{for } i = 1, \ldots, m$$

and the symmetric dominance $\Theta(\mathbf{y}') \geqq \Theta(\mathbf{y}'')$ is equivalent to (25).

The first stochastic dominance can be verified directly by comparing values of the cumulative distribution functions (23) or by comparing values of the quantile functions (25). Both the approaches require formally infinite numbers of inequalities to be verified, for all possible outcome values $v \in R$, and all possible tolerance levels $\beta \in (0,1]$, respectively. In the case of finite lotteries the quantile conditions (25) can be reduced to a finite system of inequalities. Similarly, in the case of a finite set (grid) of possible outcomes the cumulative distribution conditions (23) is reduced to a finite system of inequalities. We illustrate this with the following example.

EXAMPLE 3.2 *Let us consider a choice among three random variables A, B and C representing returns (in %) from some investments. The distributions of outcomes are given in Table 1.*

One may easily notice that all the outcomes are integers between 3 and 11. Hence, the FSD can be verified directly according to (23) by examination of

Table 1. Sample outcome distributions

A		B		C	
Outcome	Probability	Outcome	Probability	Outcome	Probability
3	0.4	5	0.1	5	0.1
4	0.3	6	0.2	7	0.1
6	0.1	8	0.1	8	0.2
7	0.1	9	0.2	9	0.2
9	0.1	10	0.4	11	0.4

Table 2. FSD verification

v	3	4	5	6	7	8	9	10	11
$F_A(v)$	0.4	0.7	0.7	0.8	0.9	0.9	1.0	1.0	1.0
$F_B(v)$	0.0	0.0	0.1	0.3	0.3	0.4	0.6	1.0	1.0
$F_C(v)$	0.0	0.0	0.1	0.1	0.2	0.4	0.6	0.6	1.0

the corresponding cumulative distribution functions at integers from 3 to 11. Table 2 contains these values. One can see that $C \succ_{FSD} B \succ_{FSD} A$. Thus the investment C is the best choice for all decision makers maximizing outcome.

The same conclusion can be derived by the analysis of quantile relations (25). Namely, all the outcome distributions can be completely characterized by quantiles corresponding to tolerance levels $i/10$ for $i = 1, \dots, 10$ (or expressed as lotteries with 10 equally probable tickets). Table 3 the quantile values (ordered achievement vectors).

Table 3. Symmetric dominance verification

i	1	2	3	4	5	6	7	8	9	10
$\theta_i(\mathbf{a}) = F_A^{(-1)}(\frac{i}{10})$	3	3	3	3	4	4	4	6	7	9
$\theta_i(\mathbf{b}) = F_B^{(-1)}(\frac{i}{10})$	5	6	6	8	9	9	10	10	10	10
$\theta_i(\mathbf{c}) = F_C^{(-1)}(\frac{i}{10})$	5	7	8	8	9	9	11	11	11	11

Two possible approaches to the FSD verification imply two corresponding multiple criteria optimization models for decisions under risk. One depends on selection of a finite set of target outcome values $v_1 < v_2 < \cdots < v_r$ and minimization of criteria $F_Y(v_k)$ for $k = 1, \dots, r$. In the case of scenarios with (possibly different) probabilities p_i such a single criterion can be expressed as:

$$\begin{aligned} F_Y(v_k) \quad &= \quad \min \sum_{i=1}^{m} p_i z_{ki} \\ &\text{s.t.} \quad v_k - y_i + \varepsilon \leq M z_{ki}, \;\; z_{ki} \in \{0,1\} \quad \text{for } i = 1, \dots, m, \end{aligned} \tag{26}$$

where ε is an arbitrarily small positive parameter while M is a sufficiently large constant.

The second approach depends on selection of a finite set of tolerance levels $0 < \beta_1 < \beta_2 < \cdots < \beta_r$ and maximization of criteria $F_Y^{(-1)}(\beta_k)$ for $k = 1, \ldots, r$. In the case of scenarios with (possibly different) probabilities p_i, a single criterion can be expressed as:

$$\begin{aligned} F_Y^{(-1)}(\beta_k) \quad = \quad & \max t_k \\ \text{s.t.} \quad & t_k - y_i \le M z_{ki}, \;\; z_{ki} \in \{0,1\} \quad \text{for } i = 1, \ldots, m \\ & \sum_{i=1}^{m} p_i z_{ki} \le \beta_k - \varepsilon, \end{aligned} \tag{27}$$

where M is a sufficiently large constant.

Both the multiple criteria models are consistent with the FSD relation. The models themselves introduce binary variables thus creating discrete structures independently of the nature of the feasible set Q. Nevertheless, the ARBDS methodology is capable to support interactive analysis of discrete problems.

4. Risk aversion and equitable efficiency

4.1. Equitable efficiency

The concept of symmetric efficiency does not limit the risk attitude of the DM. It covers both pessimistic (risk averse) and optimistic (risk seeking) preferences. In order to focus on the risk averse decision makers (Bell et al., 1988), the preference model should satisfy the Pigou–Dalton principle of transfers. The *principle of transfers* states that a transfer of small amount from an outcome to any relatively worse–off outcome results in a more preferred achievement vector, i.e.

$$y_{i'} > y_{i''} \quad \Rightarrow \quad \mathbf{y} - \varepsilon \mathbf{e}_{i'} + \varepsilon \mathbf{e}_{i''} \succ \mathbf{y} \quad \text{for } 0 < \varepsilon < y_{i'} - y_{i''}. \tag{28}$$

In terms of outcome distributions the principle of transfer depicts that any mean preserving contraction results in a more preferred (less risky) distribution. The rational preference relations satisfying the requirement of impartiality (12) and the principle of transfers (28) we will call hereafter *equitable rational preference relations.*

The equitable rational preference relations allow us to define the concept of equitably efficient solution (Kostreva and Ogryczak, 1999), similar to the Pareto efficient solution defined with the rational preference relations. We say that achievement vector $\mathbf{y}'$ *equitably dominates* $\mathbf{y}''$ ($\mathbf{y}' \succ_e \mathbf{y}''$), iff $\mathbf{y}' \succ \mathbf{y}''$ for all equitable rational preference relations $\succeq$. We say that a feasible solution $\mathbf{x} \in Q$ is *equitably efficient*, (is an equitably efficient solution of the multiple criteria problem (1)) if and only if there does not exist any $\mathbf{x}' \in Q$ such that $\mathbf{f}(\mathbf{x}') \succ_e \mathbf{f}(\mathbf{x})$. Note that each equitably efficient solution is also an efficient solution but not vice verse.

The equitable dominance relation may be illustrated with the domination structure (13). For the standard rational dominance relation, the sets $D(\mathbf{y})$

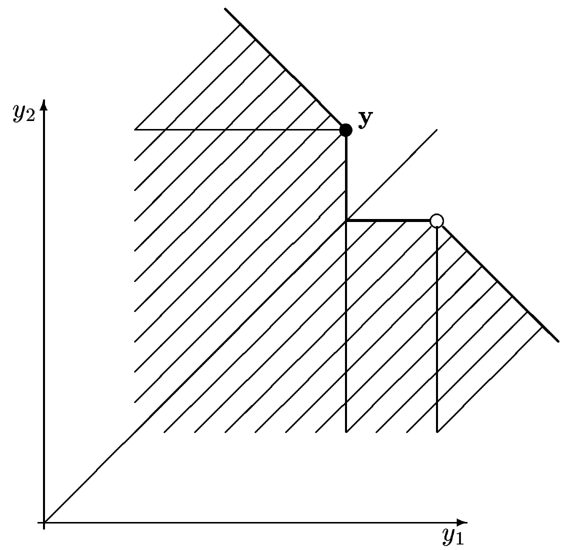

Figure 3. Equitable dominance structure in R^2

are independent of $\mathbf{y}$ and they take the form of the nonnegative orthant. The domination structure of the symmetric dominance is a union of the set defined by the nonnegative orthants for the achievement vector $\mathbf{y}$ and all its permutations (symmetric copies), as shown in Figure 1. The domination structure of the equitable dominance depends additionally on the location of an achievement vector $\mathbf{y}$ relative to the absolute equity line ($y_1 = y_2 = \cdots = y_m$). In the general case, the set $D(\mathbf{y})$ is not a cone and it is not convex. Figure 3 shows $D(\mathbf{y})$ fixed at $\mathbf{y}$, i.e. the set $\mathbf{y} + D(\mathbf{y})$. Although, when we consider directions leading to outcome vectors dominating given $\mathbf{y}$, i.e., $S(\mathbf{y}) = \{\mathbf{d} \in Y : \ \mathbf{y} + \mathbf{d} \succ \mathbf{y}\} \cup \{\mathbf{0}\}$, we get a convex set. Figure 4 shows $S(\mathbf{y})$ fixed at $\mathbf{y}$, i.e. the set $\mathbf{y} + S(\mathbf{y})$.

The relation of equitable dominance can be expressed as a vector inequality on the cumulative ordered achievement vectors. This can be mathematically formalized as follows. Recall that we have introduced the ordering map Θ such that $\theta_1(\mathbf{y}) \le \theta_2(\mathbf{y}) \le \cdots \le \theta_m(\mathbf{y})$. This allows us to focus on distributions of outcomes impartially. Next, we apply to ordered achievement vectors $\Theta(\mathbf{y})$, a linear cumulative map to get

$$\bar{\theta}_i(\mathbf{y}) = \sum_{j=1}^{i} \theta_j(\mathbf{y}) \quad \text{for } i = 1, \ldots, m. \tag{29}$$

Hence, the coefficients of vector $\bar{\Theta}(\mathbf{y})$ express, respectively: the worst (smallest) outcome, the total of the two worst outcomes, the total of the three worst outcomes, etc.

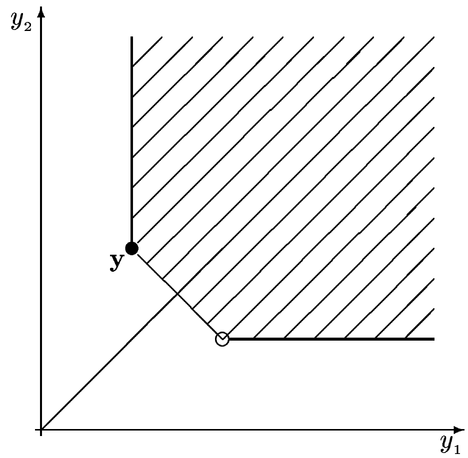

Figure 4. The set of outcomes equitably dominating $\mathbf{y} \in R^2$

Directly from the definition of $\bar{\Theta}$, it follows that for any two achievement vectors $\mathbf{y}', \mathbf{y}'' \in Y$ equation $\bar{\Theta}(\mathbf{y}') = \bar{\Theta}(\mathbf{y}'')$ holds if and only if $\mathbf{y}'$ and $\mathbf{y}''$ have the same distribution of outcomes (i.e., $\Theta(\mathbf{y}') = \Theta(\mathbf{y}'')$). Similarly, inequality $\Theta(\mathbf{y}') \geq \Theta(\mathbf{y}'')$ implies $\bar{\Theta}(\mathbf{y}') \geq \bar{\Theta}(\mathbf{y}'')$ but the reverse implication is not valid. For instance, $\bar{\Theta}(2,2,2) = (2,4,6) \geq (1,3,6) = \bar{\Theta}(1,2,3)$ and simultaneously $\Theta(2,2,2) \not\geq \Theta(1,2,3)$.

The relation $\bar{\Theta}(\mathbf{y}') \geq \bar{\Theta}(\mathbf{y}'')$ was extensively analyzed within the theory of majorization (Marshall and Olkin, 1979), where it is called the relation of weak supermajorization. The theory of majorization includes the results which allow us to derive the following assertion (Kostreva and Ogryczak, 1998).

PROPOSITION 4.1 *Achievement vector $\mathbf{y}' \in Y$ equitably dominates $\mathbf{y}'' \in Y$, if and only if $\bar{\theta}_i(\mathbf{y}') \geq \bar{\theta}_i(\mathbf{y}'')$ for all $i \in I$ where at least one strict inequality holds.*

Proposition 4.1 allows us to express equitable efficiency for problem (1) in terms of the standard efficiency for the multiple criteria problem with objectives $\bar{\Theta}(\mathbf{f}(\mathbf{x}))$:

$$\max \{(\bar{\theta}_1(\mathbf{f}(\mathbf{x})), \bar{\theta}_2(\mathbf{f}(\mathbf{x})), \ldots, \bar{\theta}_m(\mathbf{f}(\mathbf{x}))) \ : \quad \mathbf{x} \in Q\}. \tag{30}$$

A feasible solution $\mathbf{x} \in Q$ is an equitably efficient solution of the multiple criteria problem (1), if and only if it is an efficient solution of the multiple criteria problem (30).

The objective functions in a multiple criteria problem can be divided by positive constants without affecting the set of efficient solutions. For better understanding of the multiple criteria problem (30), one may consider normalized objective functions $\bar{\theta}_i(\mathbf{y})/i$ for $i = 1, \ldots, m$ thus representing averages of the i

smallest outcomes in $\mathbf{y}$. Note that the first objective $\bar{\theta}_1(\mathbf{y})/1$ represents then the worst outcome (the maximin approach (6)) and the last objective $\bar{\theta}_m(\mathbf{y})/m$ represents the mean outcome $\frac{1}{m}\sum_{i=1}^m y_i$ (the maxisum approach (5)). On the other hand, the risk seeking strategy of the maximization of the best possible outcome (the maximax approach) is not available in the multiple criteria model (30). Ogryczak (2000) has shown how various mean-risk approaches to portfolio optimization can be derived from the model (30).

The optimization formula (15) for $\theta_k(\mathbf{y})$ can easily be extended to define $\bar{\theta}_k(\mathbf{y})$. Namely, for any $k = 1, 2, \ldots, m$ the following formula is valid:

$$\begin{array}{rll} \bar{\theta}_k(\mathbf{y}) & = & \max\ kt_k - \sum_{i=1}^{m} d_{ki} \\ & \text{s.t.} & t_k - y_i \le d_{ki},\ d_{ki} \ge 0 \qquad \text{for } i = 1, \ldots, m, \\ & & d_{ki} \le M z_{ki},\ z_{ki} \in \{0, 1\} \qquad \text{for } i = 1, \ldots, m, \\ & & \sum_{i=1}^{m} z_{ki} \le k - 1. \end{array} \tag{31}$$

where M is a sufficiently large constant (larger than any possible difference between various individual outcomes y_i). However, the optimization problem defining the cumulated ordered outcome can be dramatically simplified since all the binary variables (and the related constraints) turn out to be redundant as shown in the following theorem.

THEOREM 4.1 *For any given vector* $\mathbf{y} \in R^m$*, the cumulated ordered coefficient* $\bar{\theta}_k(\mathbf{y})$ *can be found as the optimal value of the following LP problem:*

$$\begin{array}{rll} \bar{\theta}_k(\mathbf{y}) & = & \max\ kt_k - \sum_{i=1}^{m} d_{ki} \\ & s.t. & t_k - y_i \le d_{ki},\ d_{ki} \ge 0 \quad \textit{for } i = 1, \ldots, m. \end{array} \tag{32}$$

Proof. In order to prove the theorem we will show that the optimal value of problem (32) is the same as that of problem (31). First of all, let us notice that any feasible solution of (31) (when ignoring variables z_{ki}) is also feasible to problem (32). Moreover, such a solution has no more than $k-1$ positive values of variables d_{ki}. Opposite, every feasible solution of problem (32) corresponds to a feasible solution of problem (31), provided that it contains no more than than $k-1$ positive values of variables d_{ki}.

On the other hand, for any feasible solution to (32) which contains $s \ge k$ positive values of variables d_{ki} one can define an alternative feasible solution: $\tilde{t}_k = t_k - \Delta$ and $\tilde{d}_{ki} = d_{ki} - \Delta$ for $d_{ki} > 0$, where Δ is an arbitrary small positive number. For at least k positive values one gets

$$k\tilde{t}_k - \sum_{i=1}^{m} \tilde{d}_{ki} = kt_k - \sum_{i=1}^{m} d_{ki} + (s-k)\Delta \ge kt_k - \sum_{i=1}^{m} d_{ki},$$

which completes the proof. □

The entire cumulative ordered multiple criteria model (30) can be formulated as the following LP extension of the original multiple criteria problem:

$$\begin{array}{ll} \max & \left[t_1 - \sum_{i=1}^{m} d_{1i},\ 2t_2 - \sum_{i=1}^{m} d_{2i}, \ldots,\ mt_m - \sum_{i=1}^{m} d_{mi} \right] \\ \text{s.t.} & \\ & t_k - f_i(\mathbf{x}) \leq d_{ki},\ d_{ki} \geq 0 \quad \text{for } i,k = 1,\ldots,m, \\ & \mathbf{x} \in Q. \end{array} \tag{33}$$

When applied to an LP multiple criteria problem (1) the extended problem remains within the class of LP. We illustrate this with the following example of an LP model for portfolio optimization.

EXAMPLE 4.1 *Consider a simple problem of portfolio optimization. Let* $J = \{1,2,\ldots,n\}$ *denote the set of securities in which one intends to invest a capital. We assume, as usual, that for each security* $j \in J$ *there is given a vector of data* $(c_{ij})_{i=1,\ldots,m}$ *(hereafter referred to as outcome), where* c_{ij} *is the observed (or forecasted) rate of return of security* j *under scenario* i *(Zenios, 1995). We consider discrete distributions of returns defined by the finite set* $I = \{1,2,\ldots,m\}$ *of equally probable scenarios. The outcome data forms an* $m \times n$ *matrix* $\mathbf{C} = (c_{ij})_{i=1,\ldots,m;j=1,\ldots,n}$ *which columns correspond to securities while rows* $\mathbf{c}_i = (c_{ij})_{j=1,2,\ldots,n}$ *correspond to outcomes. Further, let* $\mathbf{x} = (x_j)_{j=1,2,\ldots,n}$ *denote the vector of decision variables defining a portfolio. Each variable* x_j *expresses the portion of the capital invested in the corresponding security. Portfolio* $\mathbf{x}$ *generates the outcomes*

$$\mathbf{y} = \mathbf{C}\mathbf{x} = (\mathbf{c}_1\mathbf{x}, \mathbf{c}_2\mathbf{x}, \ldots, \mathbf{c}_m\mathbf{x}),$$

representing the portfolio returns under several scenarios. The portfolio selection problem can be considered as an LP optimization problem with m *uniform objective functions* $f_i(\mathbf{x}) = \mathbf{c}_i\mathbf{x} = \sum_{j=1}^{n} c_{ij}x_j$ *to be maximized (Ogryczak, 2000):*

$$\max \{\mathbf{C}\mathbf{x} \ : \ \sum_{j=1}^{n} x_j = 1, \quad x_j \geq 0 \quad \textit{for } j = 1,\ldots,n\}.$$

Typical investors represent risk averse preferences. Hence, our portfolio optimization problem can be considered a case of the multiple criteria problem (1) with the equitably rational preference model. Due to (33), we can formulate the *portfolio selection problem as the following multiple criteria LP model:*

$$\begin{array}{lll} \max & [\delta_1,\ \delta_2, \ldots, \delta_m] & \\ s.t. & & \\ & \delta_k = kt_1 - \sum_{i=1}^{m} d_{ki} & \textit{for } k = 1, \ldots, m, \\ & t_k - \sum_{j=1}^{n} c_{ij} x_j \leq d_{ki},\ d_{ki} \geq 0 & \textit{for } i, k = 1, \ldots, m, \\ & \sum_{j=1}^{n} x_j = 1 \textit{ and } x_j \geq 0 & \textit{for } j = 1, \ldots, n, \end{array}$$

covering all the rational risk averse preferences.

Recall Example 3.1 showing a not connected symmetrically efficient set for a bicriteria LP problem. Figure 2 shows the standard efficient set consisting of two line segments $\overline{\mathrm{AB}}$ and $\overline{\mathrm{BC}}$, where A=(0,21), B=(9,18) and C=(23.4,0). While taking into account symmetric dominance we have eliminated segments $\overline{\mathrm{AD}}$ and $\overline{\mathrm{EF}}$, thus resulting in the symmetric efficient set consisted of three segments: $\overline{\mathrm{DB}}$, $\overline{\mathrm{BE}}$ and $\overline{\mathrm{FC}}$. Further, taking into account the equitable dominance we get the segment $\overline{\mathrm{BE}}$ as the equitably efficient set. Actually, the equitably efficient sets for multiple criteria LP problems are always connected (Kostreva and Ogryczak, 1999).

4.2. Distribution approach

Consider again the case when there exists a finite set of all possible outcomes of the individual objective functions and we can directly deal with the distribution of outcomes described by frequencies of several outcomes. Let $V = \{v_1, v_2, \ldots, v_r\}$ (where $v_1 < v_2 < \cdots < v_r$) denote the set of all attainable outcomes (all possible values of the individual objective functions f_i for $\mathbf{x} \in Q$). Recall that we have introduced integer functions $h_k(\mathbf{y})$ $(k = 1, \ldots, r)$ expressing the number of values v_k taken in the achievement vector $\mathbf{y}$ as well as the cumulative distribution functions $\bar{h}_k(\mathbf{y})$ expressing the number of outcomes smaller or equal to v_k. In order to take into account the principle of transfers we need to distinguish values of outcomes smaller or equal to v_k. For this purpose we weight vector $\bar{\mathbf{h}}(\mathbf{y})$ to get:

$$\hat{h}_1(\mathbf{y}) = 0 \quad \text{and} \quad \hat{h}_k(\mathbf{y}) = \sum_{l=1}^{k-1} (v_{l+1} - v_l)\bar{h}_l(\mathbf{y}) \quad \text{for } k = 2, \ldots, r. \tag{34}$$

Note that

$$\begin{aligned}\hat{h}_k(\mathbf{y}) &= \sum_{l=1}^{k-1} [(v_{l+1} - v_l) \sum_{j=1}^{l} h_j(\mathbf{y})] \\ &= \sum_{l=1}^{k-1} (v_k - v_l) h_l(\mathbf{y}) \quad \text{for } k = 2, \ldots, r.\end{aligned} \tag{35}$$

In other words, $\hat{h}_k(\mathbf{y})$ expresses the total of differences between v_k and all the outcomes y_i smaller than v_k. Since $(v_k - v_l) > 0$ for $1 \leq l < k$, it follows from (35) that vector function $\hat{\mathbf{h}}(\mathbf{y})$ provides a unique description of the distribution of coefficients of vector $\mathbf{y}$, i.e., for any $\mathbf{y}', \mathbf{y}'' \in V^m$ one gets:

$$\hat{\mathbf{h}}(\mathbf{y}') = \hat{\mathbf{h}}(\mathbf{y}'') \quad \Leftrightarrow \quad \mathbf{h}(\mathbf{y}') = \mathbf{h}(\mathbf{y}'') \quad \Leftrightarrow \quad \Theta(\mathbf{y}') = \Theta(\mathbf{y}'').$$

Moreover the following assertion is valid (Ogryczak, 1997). For achievement vectors $\mathbf{y}', \mathbf{y}'' \in V^m$,

$$\hat{\mathbf{h}}(\mathbf{y}') \leq \hat{\mathbf{h}}(\mathbf{y}'') \quad \Leftrightarrow \quad \bar{\Theta}(\mathbf{y}') \geq \bar{\Theta}(\mathbf{y}''). \tag{36}$$

Equivalence (36) permits one to express equitable efficiency for problem (1) in terms of the standard efficiency for the multiple criteria problem with objectives $\hat{\mathbf{h}}(\mathbf{f}(\mathbf{x}))$:

$$\min \{(\hat{h}_1(\mathbf{f}(\mathbf{x})), \hat{h}_2(\mathbf{f}(\mathbf{x})), \ldots, \hat{h}_r(\mathbf{f}(\mathbf{x}))) \ : \quad \mathbf{x} \in Q\}. \tag{37}$$

PROPOSITION 4.2 *A feasible solution* $\mathbf{x} \in Q$ *is an equitably efficient solution of the multiple criteria problem (1), if and only if it is an efficient solution of the multiple criteria problem (37).*

Note that $\hat{h}_1(\mathbf{y}) = 0$ for any $\mathbf{y}$ which means that the first criterion is constant and redundant in problem (37). Moreover, $mv_r - \hat{h}_r(\mathbf{y}) = \sum_{i=1}^{m} y_i$. Thus single objective minimization of the last criterion in problem (37) is equivalent to maximization of the sum of all the original criteria in problem (1). In other words, the maximization of the mean (expected) outcome is equivalent to a single criterion minimization in the multiple criteria optimization problem (37):

$$E[Y] = \frac{1}{m} \sum_{i=1}^{m} y_i = v_r - \frac{1}{m} \hat{h}_r(\mathbf{y}).$$

One may notice that for any utility function u the corresponding expected utility (Levy, 1992) may be expressed as a linear combination of criteria $\hat{h}_r(\mathbf{y})$. Namely, it follows from (21) and (34) that:

$$E[u(Y)] = \frac{1}{m} \sum_{i=1}^{m} u(y_i) = u(v_r) + \sum_{k=2}^{r} \frac{w_k}{m} \bar{h}_k(\mathbf{y}),$$

where the weights w_k are given as:

$$w_k = \left[\frac{u(v_{k+1}) - u(v_k)}{v_{k+1} - v_k} - \frac{u(v_k) - u(v_{k-1})}{v_k - v_{k-1}}\right] \quad \text{for } k = 2, \ldots, r-1,$$

$$w_r = -\frac{u(v_r) - u(v_{r-1})}{v_r - v_{r-1}}.$$

Certainly, in the case of a strictly increasing and concave utility function u, all the weight coefficients w_k are negative. Therefore, maximization of the expected utility generates then an efficient solution of problem (37) and consequently an equitably efficient solution to the original multiple criteria problem (1). However, as based on the weighting approach (9), the expected utility approach does not allows us to identify all the equitably efficient solutions.

Formula (35) allows us to express $\hat{h}_k(\mathbf{y})$ as a piece wise linear function of $\mathbf{y}$:

$$\hat{h}_k(\mathbf{y}) = \sum_{i=1}^{m} (v_k - y_i)_+ = \sum_{i=1}^{m} \max\{v_k - y_i, 0\} \quad \text{for } k = 1, \ldots, r. \tag{38}$$

Hence, the quantity $\hat{h}_k(\mathbf{y})$ can be computed directly by the minimization:

$$\begin{aligned} \hat{h}_k(\mathbf{y}) \quad &= \quad \min \sum_{i=1}^{m} t_{ki} \\ &\text{s.t.} \quad v_k - y_i \le t_{ki}, \; t_{ki} \ge 0 \quad \text{for } i = 1, \ldots, m. \end{aligned} \tag{39}$$

Therefore, the entire multiple criteria model (37) can be formulated as follows:

$$\begin{aligned} \min \quad & \left[\sum_{i=1}^{m} t_{2i}, \; \sum_{i=1}^{m} t_{3i}, \ldots, \; \sum_{i=1}^{m} t_{ri}\right] \\ \text{s.t.} \quad & \\ & v_k - f_i(\mathbf{x}) \le t_{ki}, \; t_{ki} \ge 0 \quad \text{for } i = 1, \ldots, m; \; k = 2, \ldots, r, \\ & \mathbf{x} \in Q \end{aligned} \tag{40}$$

Note that opposite to problem (20), the above formulation does not use integer variables and can be considered as an LP modification of the original multiple criteria problem (1).

4.3. SSD and general MCO models

Formula (38) can easily be extended for any target values $v \in R$ to define the quantity $\hat{h}_v(\mathbf{y})$ as follows:

$$\hat{h}_v(\mathbf{y}) = \sum_{i=1}^{m} (v - y_i)_+ \quad \text{for } v \in R. \tag{41}$$

While $\frac{1}{m}\bar{h}_v(\mathbf{y})$ represents the right continuous cumulative distribution function of the random variable Y generating outcomes y_i (see (22)), $\frac{1}{m}\hat{h}_v(\mathbf{y})$ is the expected shortfall to the target v and therefore (Ogryczak and Ruszczyński, 1999; 2001) the corresponding integral of F_Y:

$$\frac{1}{m}\hat{h}_v(\mathbf{y}) = E\{(v - Y)_+\} = F_Y^{(2)}(v) = \int_{-\infty}^{v} F_Y(\xi)d\xi. \tag{42}$$

Hence, inequality $\hat{\mathbf{h}}(\mathbf{y}') \leq \hat{\mathbf{h}}(\mathbf{y}'')$ represents the relation of the *second order stochastic dominance (SSD)* (Levy, 1992). Recall that the weak SSD dominance is defined as

$$Y' \succeq_{SSD} Y'' \quad \Leftrightarrow \quad F_{Y'}^{(2)}(v) \leq F_{Y''}^{(2)}(v) \quad \forall v \in R. \tag{43}$$

We say that random variable Y' dominates Y'' according to SSD if $F_{Y'}^{(2)}(v) \leq F_{Y''}^{(2)}(v)$ for any $v \in R$, and there exists $v_0 \in R$ such that a strict inequality holds.

The SSD relation is crucial for decision making under risk. If $R_{\mathbf{x}'} \succ_{SSD} R_{\mathbf{x}''}$, then $R_{\mathbf{x}'}$ is preferred to $R_{\mathbf{x}''}$ within all risk-averse preference models where larger outcomes are preferred (Whitmore and Findlay, 1978). In a general case it is based on the comparison of the infinite number (continuum) of criteria defined as values of the the expected shortfalls $F^{(2)}(v)$ for every possible real target $v \in R$.

To obtain a quantile representation of the SSD we introduce the *second quantile function* defined for a random variable Y as:

$$F_Y^{(-2)}(p) = \int_0^p F_Y^{(-1)}(\alpha)d\alpha \quad \text{for } 0 < p \leq 1 \quad \text{and} \quad F_Y^{(-2)}(0) = 0. \tag{44}$$

Similarly to $F_Y^{(2)}$, the function $F_Y^{(-2)}$ is convex. The graph of $F_Y^{(-2)}$ is called the *absolute Lorenz curve*. The pointwise comparison of the second quantile functions defines the so-called absolute (or general) Lorenz order (Shorrocks, 1983).

Recently, an intriguing duality relation between the second quantile function $F_Y^{(-2)}$ and the second performance function $F_Y^{(2)}$ has been shown (Ogryczak and Ruszczyński, 2002). Namely, function $F_Y^{(-2)}$ is a conjugent (Rockafellar, 1970) of $F_Y^{(2)}$, i.e., for every $p \in [0, 1]$, one gets

$$F_Y^{(-2)}(p) = \sup_{\eta}\{\eta p - F_Y^{(2)}(\eta)\}. \tag{45}$$

It follows from the duality theory that we may fully characterize the SSD relation by using the conjugate function $F^{(-2)}$:

$$Y' \succeq_{SSD} Y'' \quad \Leftrightarrow \quad F_{Y'}^{(-2)}(p) \geq F_{Y''}^{(-2)}(p) \quad \text{for all } 0 \leq p \leq 1. \tag{46}$$

In other words, the absolute Lorenz order is equivalent to the SSD order.

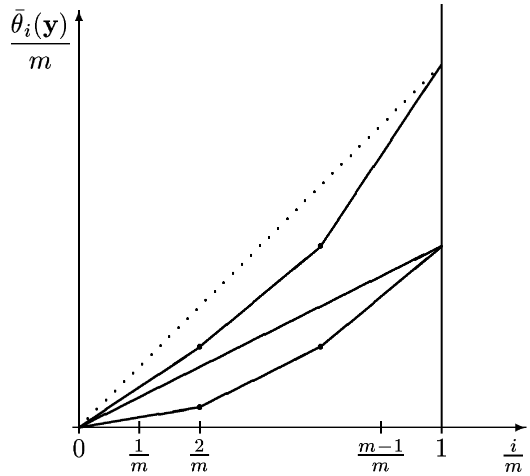

Figure 5. $\bar{\Theta}(\mathbf{y})$ and the absolute Lorenz curves.

If random variable Y represents a lottery $\mathbf{y}$ with m equally probable tickets, then

$$\frac{1}{m}\bar{\theta}_i(\mathbf{y}) = F_Y^{(-2)}(\frac{i}{m}) \quad \text{for } i = 1, \ldots, m$$

and the equitable dominance $\bar{\Theta}(\mathbf{y}') \geqq \bar{\Theta}(\mathbf{y}'')$ is equivalent to (46). Vector $\bar{\Theta}(\mathbf{y})$ can be viewed graphically with the absolute Lorenz curve (Fig. 5) connecting point (0,0) and points $(i/m, \bar{\theta}_i(\mathbf{y})/m)$ for $i = 1, \ldots, m$. Note that the construction of the absolute Lorenz curves is then similar to the standard Lorenz curve (Marshall and Olkin, 1979) for the population of m outcomes. However, the standard Lorenz curves are considered for positive outcomes and normalized by their mean. Therefore, in terms of the Lorenz curves no achievement vector can be better than the vector of equal outcomes. Comparison of absolute Lorenz curves takes into account also values of outcomes. Vectors of equal outcomes are distinguished according to the value of their outcomes. They are graphically represented with various ascent lines in Fig. 5. With the equitable dominance, an achievement vector of large unequal outcomes may dominate an achievement vector with small equal outcomes.

The second stochastic dominance can be verified directly by comparing values of the second cumulative distribution functions (43) or by comparing values of the quantile functions (46). Both the approaches require formally infinite numbers of inequalities to be verified, for all possible outcome values $v \in R$, and all possible tolerance levels $\beta \in (0, 1]$, respectively. In the case of finite lotteries the quantile conditions (46) can be reduced to a finite system of inequalities. Similarly, in the case of a finite set (grid) of possible outcomes the cumulative distribution conditions (46) is reduced to a finite system of inequalities. We illustrate this with the following example.

EXAMPLE 4.2 *Let us consider a choice among three random variables A, B and C representing returns (in %) from some investments. The distributions of outcomes are given in Table 4.*

Table 4. Sample outcome distributions

A		B		C	
Outcome	Probability	Outcome	Probability	Outcome	Probability
4	0.2	5	0.1	6	0.4
6	0.3	6	0.3	7	0.3
8	0.4	7	0.2	8	0.2
10	0.1	8	0.3	10	0.1
		9	0.1		

Table 5. SSD verification

v	4	5	6	7	8	9	10
$F_A(v)$	0.2	0.2	0.5	0.5	0.9	0.9	1.0
$F_B(v)$	0.0	0.1	0.4	0.6	0.9	1.0	1.0
$F_C(v)$	0.0	0.0	0.4	0.7	0.9	0.9	1.0
$F_A^{(2)}(v)$	0.0	0.2	0.4	0.9	1.4	2.3	3.2
$F_B^{(2)}(v)$	0.0	0.0	0.1	0.5	1.1	2.0	3.0
$F_C^{(2)}(v)$	0.0	0.0	0.0	0.4	1.1	2.0	2.9

All the possible outcomes are integers between 4 and 10. Hence, for the FSD verification one may examine the corresponding cumulative distribution functions at integers from 4 to 10. Table 5 contains these values. One can see that there is no FSD relation among the investment opportunities. The same conclusion can be derived by examination of the inverse cumulative distributions (equivalent lotteries) as given in Table 6.

Table 6. Equitable dominance verification

i	1	2	3	4	5	6	7	8	9	10
$\theta_i(\mathbf{a}) = F_A^{(-1)}(\frac{i}{10})$	4	4	6	6	6	8	8	8	8	10
$\theta_i(\mathbf{b}) = F_B^{(-1)}(\frac{i}{10})$	5	6	6	6	7	7	8	8	8	9
$\theta_i(\mathbf{c}) = F_C^{(-1)}(\frac{i}{10})$	6	6	6	6	7	7	7	8	8	10
$\frac{1}{10}\bar{\theta}_i(\mathbf{a}) = F_A^{(-2)}(\frac{i}{10})$	0.4	0.8	1.4	2.0	2.6	3.4	4.2	5.0	5.8	6.8
$\frac{1}{10}\bar{\theta}_i(\mathbf{b}) = F_B^{(-2)}(\frac{i}{10})$	0.5	1.1	1.7	2.3	3.0	3.7	4.5	5.3	6.1	7.0
$\frac{1}{10}\bar{\theta}_i(\mathbf{c}) = F_C^{(-2)}(\frac{i}{10})$	0.6	1.2	1.8	2.4	3.1	3.8	4.5	5.3	6.1	7.1

In order to examine if the SSD relation occurs among the investment opportunities we compute the cumulative ordered quantiles $\bar{\Theta}$ (absolute Lorenz curves) as in Table 6. One may notice that $C \succ_{SSD} B \succ_{SSD} A$. Hence, the investment C

is better under all risk averse preference models. The same conclusion can be derived by examination of the cumulative distribution values $F_Y^{(2)}(v_k) = \hat{h}_k(\mathbf{y})/m$ *presented in Table 5. As in our case* $v_1 = 4$, $v_2 = 5$, $\ldots$, $v_7 = 10$ *all the differences* $v_l - v_{l-1}$ *are constant and equal to 1, the quantities* $\hat{h}_k(\mathbf{y})/m$ *are easily generated by a direct cumulation of values from Table 5.*

Two dual approaches to the SSD verification imply two possible multiple criteria optimization models for risk aversion preferences. One depends on selection of a finite set of target outcome values $v_1 < v_2 < \cdots < v_r$ and minimization of criteria $F_Y^{(2)}(v_k)$ for $k = 1, \ldots, r$. This approach may be very attractive with fuzzy (Zimmermann, 1996) modeling of target values. In the case of scenarios with (possibly different) probabilities p_i, following the formula $F_Y^{(2)}(v) = E\{(v - Y)_+\}$, such a model can be expressed as:

$$\begin{array}{lll} \min & [\tau_1,\ \tau_2, \ldots, \tau_r] & \\ \text{s.t.} & & \\ & \tau_k = \sum_{i=1}^{m} p_i t_{ki} & \text{for } k = 1, \ldots, r \\ & v_k - f_i(\mathbf{x}) \le t_{ki},\ t_{ki} \ge 0 & \text{for } i = 1, \ldots, m;\ k = 1, \ldots, r \\ & \mathbf{x} \in Q & \end{array} \tag{47}$$

The second approach depends on selection of a finite set of tolerance levels $0 < \beta_1 < \beta_2 < \cdots < \beta_r \le 1$ and maximization of criteria $F_Y^{(-2)}(\beta_k)$ for $k = 1, \ldots, r$. This approach is very attractive when one needs to emphasize the risk of extreme events (Haimes, 1993). In the case of scenarios with (possibly different) probabilities p_i, due to (45), such a multiple criteria model can be expressed as:

$$\begin{array}{lll} \max & [\delta_1,\ \delta_2, \ldots, \delta_r] & \\ \text{s.t.} & & \\ & \delta_k = \beta_k t_k - \sum_{i=1}^{m} p_i d_{ki} & \text{for } k = 1, \ldots, r \\ & t_k - f_i(\mathbf{x}) \le d_{ki},\ d_{ki} \ge 0 & \text{for } i = 1, \ldots, m;\ k = 1, \ldots, r \\ & \mathbf{x} \in Q & \end{array} \tag{48}$$

Both the multiple criteria models are consistent with the SSD relation in the sense that the dominance of two distributions of outcomes $Y' \succeq_{SSD} Y''$ implies $\tau'_k \le \tau''_k$ and $\delta'_k \ge \delta''_k$ for all $k = 1, \ldots, r$, respectively. Moreover, the models result in LP expansions of the original decision problem thus offering, for instance, multiple criteria LP models for portfolio optimization. The models open up an opportunity to apply various interactive techniques of multiple criteria decision support. In particular, the ARBDS methodology can be used to support decisions under risk while preserving the risk averse preference model.

5. Concluding remarks

Multiple criteria optimization is closely related to the theory of decisions under uncertainty. Most of the classical solution concepts commonly used in the multiple criteria optimization have their origins in some approaches to handle uncertainty in the decision analysis. Actually, as shown in the paper, the optimization of multiple independent criteria and the optimization of a scalar uncertain outcome with various realizations under several scenarios are equivalent equivalent problems since the Pareto efficiency concept meets both the corresponding preference models.

In this paper we have shown how that the decisions under risk, and specifically the risk aversion preferences, can be effectively modeled with the multiple criteria optimization methodology. Multiple criteria models for general problems of decision under risk may require the use of auxiliary integer variables since they allow for possible risk seeking preferences. The risk averse preferences can be modeled by the use linear programming techniques thus leading to very simple multiple criteria models. The models provide a methodological basis allowing to take advantages of the interactive multiple criteria techniques for the process of decision support under risk. In particular, interactive techniques of the reference point method can be effectively applied to linear as well as mixed integer models.

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