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Włodzimierz Ogryczak, Michał Pióro and Artur Tomaszewski

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Warsaw University of Technology
Institute of Control & Computation Engineering
ul. Nowowiejska 15/19, 00–665 Warsaw, Poland
phone (48-22)-660-7397, fax (48-22)-825-3719

Telecommunications Network Design and Max-Min Optimization Problem

Włodzimierz Ogryczak* — *ogryczak@ia.pw.edu.pl*
Michał Pióro† — *mpp@tele.pw.edu.pl*,
Artur Tomaszewski‡ — *artur@tele.pw.edu.pl*,

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Abstract

Telecommunications networks are facing increasing demand for Internet services. Therefore, the problem of telecommunications network design with the objective to maximize service data flows and provide fair treatment of all services is very up-to-date. In this application, the so-called Max-Min Fair (MMF) solution concept is widely used to formulate the resource allocation scheme. It assumes that the worst service performance is maximized and the solution is additionally regularized with the lexicographic maximization of the second worst performance, the third one etc. In this paper we discuss solution algorithms for MMF problems related to telecommunications network design. Due to lexicographic maximization of ordered quantities, the MMF solution concept cannot be tackled by the standard optimization model (mathematical programme). However, one can formulate a sequential lexicographic optimization procedure. The basic procedure is applicable only for convex models, thus it allows to deal with basic design problems but fails if practical discrete restrictions commonly arriving in telecommunications network design are to be taken into account. Then, however, alternative sequential approaches allowing to solve non-convex MMF problems can be used. They depend on replacement of the original problem with the lexicographic minimization of the vector that describes the distribution of outcome values, which, fortunately enough, is convex as long as an original problem is defined with a convex feasible set and a concave objective functions.

Key words. Telecommunications, network design, resource allocation, fairness, lexicographic optimization, lexicographic max-min

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1 Introduction

Since the emergence of the Internet one has witnessed an unprecedented growth of traffic that is carried in the telecommunications networks. The pace at which the number of network users and the amount of traffic related to data-oriented applications are growing has been and still is much higher than several percent of growth that were typical for traditional voice-only networks; as a matter of fact data traffic almost doubles every year. It can also be observed that the distribution of traffic in data networks changes quickly, both - in the short and long time-scales, and is very difficult to predict. As a result, from the network operator's perspective the network extension process becomes very complicated - while it is not economically feasible to sufficiently over-dimension a network, it is also hard to decide when and where the network should be augmented. An inevitable effect of the situation that the capacity of a network does not match the traffic generated by network service users, is network overload - a phenomenon commonly encountered in current data-oriented networks.

Overloads influence the quality of service perceived by users - data transfer slows down because packet transfer delays increase and packet losses occur much more frequently. Overloads are one of the major concerns of network operators, because the guaranteed quality of service level is one of the basic elements of network operators' differentiation and a prerequisite of their success. In order to avoid overloads and provide the guaranteed quality of service level (instead of offering the so-called best-effort service) the network operator must control the amount of traffic that enters the network. The traffic admission control process is responsible for deciding how many users can be served and how much traffic each of these users can generate. What is important is that, in general, some users will be denied the service in order to reduce the overall stream of traffic that enters the network. Since the service denial probability is another important measure of the quality of service level, one of the primary objectives of the admission control process must be to guarantee that the users have fair access to network services. The most common "fairness-oriented" (as opposed to "revenue-oriented") approach is to admit equal amount of traffic from every stream - the amount being expressed in absolute or relative terms. Unfortunately, this approach can result in poor network capacity utilization, since for many streams much more traffic could still be admitted than this actual amount. Thus, one of the alternative approaches is to admit as much traffic as possible from every stream while making the smaller admitted amounts as large as possible.

The problem to determine how much traffic of every traffic stream should be admitted into the network, and how the admitted traffic should be routed through the network so as to satisfy the requirements of high network utilization and to guarantee fairness to the users, is one of the most challenging problems of current telecommunications networks design. In this paper we show how this problem is related to two well known OR problems - namely the max-min optimization problem and the lexicographic optimization problem. We study the general formulations of these problems and analyze how to use their notions to express the fairness of the traffic admission process. We go on to formulate basic network design problems and study the complexity of the obtained formulations. We analyze the methods of max-min and lexicographic optimization and examine how they can be applied to solve the presented network design problem.

The paper is organized as follows. In Section 2 we introduce the lexicographic Max-Min or the Max-Min Fair (MMF) solution concept and summarize its major properties.

In Section 3 we present details of three telecommunications problems leading to MMF formulations. Further in Section 4 we discuss solution algorithms for the lexicographic Max-Min optimization and analyze their applicability for telecommunications problems.

2 Max-Min and the MMF solution concept

2.1 Max-Min solution concepts

The problem we consider may be viewed in terms of resource allocation decisions as follows. Let us assume there is a set of m services. There is also a set Q of resource allocation patterns (allocation decisions). For each service j a function $f_j(\mathbf{x})$ of allocation pattern $\mathbf{x}$ has been defined. This function, called the individual objective function, measures the outcome (effect) $y_j = f_j(\mathbf{x})$ of the allocation pattern for service j. The outcomes can be measured (modeled) as service quality, service amount, service time, service costs as well as in a more subjective way the (client's) utility of the provided service. In typical formulations a greater value of the outcome means a better effect (higher service quality or client satisfaction); otherwise, the outcomes can be replaced with their complements to some large number. Therefore, without loss of generality, we can assume that each individual outcome y_j is to be maximized which results in a multiple criteria maximization model. The problem can be formulated as follows:

$$\max \{\mathbf{f}(\mathbf{x}) : \mathbf{x} \in Q\} \tag{2.1}$$

where $Q \subseteq \Re^n$ is a feasible set and $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), \dots, f_m(\mathbf{x}))$ is a vector of real-valued functions $f_j : Q \to \Re, j = 1, 2, \dots, m$, where $\mathbf{x} = (x_1, x_2, \dots, x_n)$ is an n-vector. We refer to the elements of the criterion space as outcome vectors. An outcome vector $\mathbf{y}$ is attainable if it expresses outcomes of a feasible solution $\mathbf{x} \in Q$ (i.e., $\mathbf{y} = \mathbf{f}(\mathbf{x})$). The set of all the attainable outcome vectors is denoted by Y. Note that, in general, convex feasible set Q and concave function $\mathbf{f}$ do not guarantee convexity of the corresponding attainable set Y. Nevertheless, the multiple criteria maximization model (2.1) can be rewritten in the equivalent form

$$\max \{\mathbf{y} : y_j \leq f_j(\mathbf{x})\ \forall j,\ \mathbf{x} \in Q\} \tag{2.2}$$

where the attainable set Y is convex whenever Q is convex and functions f_j are concave. Model (2.1) only specifies that we are interested in maximization of all objective functions f_j for $j \in M = \{1, 2, \dots, m\}$. Each attainable outcome vector $\mathbf{y} \in Y$ is called *nondominated* if one cannot improve any individual outcome without worsening another one. Each feasible solution $\mathbf{x} \in Q$ generating the nondominated outcome is called an *efficient* (Pareto-optimal) solution of the multiple criteria problem (2.1). In other words, a feasible solution for which one cannot improve any outcome without worsening another is efficient [33]. In order to make model (2.1) operational, one needs to assume some solution concept specifying what it means to maximize multiple objective functions. Simple solution concepts are defined by achievement functions $\theta : Y \to \Re$ to be maximized. Thus the multiple criteria problem (2.1) is replaced with the aggregation $\max \{\theta(\mathbf{f}(\mathbf{x})) :\ \mathbf{x} \in Q\}$. The most commonly used achievement function is the mean (or simply the sum) of individual performances; this defines the so-called maxsum solution concept. This solution concept is primarily concerned with the overall system efficiency. As based on averaging, it often provides a solution where some services are discriminated in terms of performances.

An alternative approach depends on the so-called Max-Min solution concept, where the worst performance is maximized:

$$\max\{\min_{j=1,\ldots,m} f_j(\mathbf{x}) \;:\; \mathbf{x} \in Q\ \}. \tag{2.3}$$

The Max-Min solution concept has been widely studied in the multi-criteria optimization methodology [33, 35]. The optimal set of the Max-Min problem (2.3) always contains an efficient solution of the original multiple criteria problem (2.1). Thus, if unique, the optimal Max-Min solution is efficient. In the case of multiple optimal solutions, one of them is efficient but also some of them may not be efficient. It is a serious flaw since practical large problems usually have multiple optimal solutions and typical optimization solvers generate one of them (essentially at random). Therefore, some additional regularization is needed to overcome this flaw of the Max-Min scalarization.
The Max-Min solution concept is regarded as maintaining equity. Indeed, in the case of a simplified resource allocation problem, the Max-Min solution

$$\max\{\min_{j=1,\ldots,m} y_j \;:\; \sum_{j=1}^{m} y_j \leq b\ \} \tag{2.4}$$

takes the form $\bar{y}_j = b/m$ for all $j \in M$ thus meeting the perfect equity requirement $\bar{y}_1 = \bar{y}_2 = \ldots = \bar{y}_m$. In the general case, with possibly more complex feasible set structure, this property is not fulfilled [23]. Nevertheless, the following assertion is valid.

Proposition 1 *If there exists a nondominated outcome vector $\bar{\mathbf{y}} \in Y$ satisfying the perfect equity requirement $\bar{y}_1 = \bar{y}_2 = \ldots = \bar{y}_m$, then $\bar{\mathbf{y}}$ is the unique optimal solution of the Max-Min problem*

$$\max\{\min_{j=1,\ldots,m} y_j \;:\; \mathbf{y} \in Y\ \}. \tag{2.5}$$

Proof. Let $\bar{\mathbf{y}} \in Y$ be a nondominated outcome vector satisfying the perfect equity requirement. This means, there exists a number α such that $\bar{y}_j = \alpha$ for $j = 1, 2, \ldots, m$. Let $\mathbf{y} \in Y$ be an optimal solution of the Max-Min problem (2.5). Suppose, there exists some index j_0 such that $y_{j_0} \neq \bar{y}_{j_0}$. Due to the optimality of $\mathbf{y}$, we have:

$$y_j \geq \min_{1\leq i\leq m} y_i \geq \min_{1\leq i\leq m} \bar{y}_i = \alpha = \bar{y}_j \quad \forall\ j = 1, \ldots, m$$

which together with $y_{j_0} \neq \bar{y}_{j_0}$ contradicts the assumption that $\bar{\mathbf{y}}$ is nondominated. ■

According to Proposition 1, the perfectly equilibrated outcome vector is a unique optimal solution of the Max-Min problem if one cannot improve any of its individual outcome without worsening some others. Unfortunately, it is not a common case and, in general, the optimal set to the Max-Min aggregation (2.3) may contain numerous alternative solutions including dominated ones. While using standard algorithmic tools to identify the Max-Min solution, one of many solutions is then selected randomly.
Actually, the distribution of outcomes may make the Max-Min criterion partially passive when one specific outcome is relatively very small for all the solutions. For instance, while allocating clients to service facilities, such a situation may be caused by existence of an isolated client located at a considerable distance from all the location of facilities.

Maximization of the worst service performances (equivalent to minimization of the maximum distance) is then reduced to maximization of the service performances for that single isolated client leaving other allocation decisions unoptimized. This is a clear case of inefficient solution where one may still improve other outcomes while maintaining fairness by leaving at its best possible value the worst outcome. The Max-Min solution may be then regularized according to the Rawlsian principle of justice. Rawls [30] considers the problem of ranking different "social states"which are different ways in which a society might be organized taking into account the welfare of each individual in each society, measured on a single numerical scale [30, p. 62]. Applying the Rawlsian approach, any two states should be ranked according to the accessibility levels of the least well–off individuals in those states; if the comparison yields a tie, the accessibility levels of the next–least well–off individuals should be considered, and so on. Formalization of this concept leads us to the lexicographic Max-Min concepts.

The lexicographic Max-Min solution is known in the game theory as the nucleolus of a matrix game. It originates from an idea, presented by Dresher [7], to select from the optimal (Max-Min) strategy set of a player a subset of optimal strategies which exploit mistakes of the opponent optimally. It has been later refined to the formal nucleolus definition [32] and generalized to an arbitrary number of objective functions [29]. The concept was early considered in the Tschebyscheff approximation [31] as a refinement taking into account the second largest deviation, the third one and further to be hierarchically minimized. Similar refinement of the fuzzy set operations has been recently analyzed [8]. Within the telecommunications or network applications the lexicographic Max-Min approach has appeared already in [11, 3] and now under the name Max-Min Fair (MMF) is treated as one of the standard fairness concepts. The approach has been used for general linear programming multiple criteria problems [1, 17], as well as for specialized problems related to (multiperiod) resource allocation [12, 16]. In discrete optimization it has been considered for various problems [4, 5] including the location-allocation ones [21].

2.2 Lexicographic optimization and MMF

Typical solution concepts for the multiple criteria problems are based on the use of aggregated achievement functions $\theta : Y \rightarrow \Re$ to be maximized, thus ranking the outcomes according to a complete preorder

$$\mathbf{y}' \succeq_\theta \mathbf{y}'' \quad \Leftrightarrow \quad \theta(\mathbf{y}') \geq \theta(\mathbf{y}''). \tag{2.6}$$

This allows one to replace the multiple criteria problem (2.1) with the standard maximization problem $\max\ \{\theta(\mathbf{f}(\mathbf{x})) :\ \mathbf{x} \in Q\}$. However, there are also well defined solution concepts which do not introduce directly any scalar measure, despite they rank the outcome vectors with a complete preorder. Especially, the lexicographic (preemptive) order is used for this purpose.

Let $\mathbf{a} = (a_1, a_2, \ldots, a_m)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_m)$ be two m-vectors. Vector $\mathbf{a}$ is lexicographically greater than vector $\mathbf{b}$, $\mathbf{a} >_{_{lex}} \mathbf{b}$, if there exists index k, $0 \leq k < m$, such that $a_j = b_j$ for all $j \leq k$ and $a_{k+1} > b_{k+1}$. Consequently, $\mathbf{a}$ is lexicographically greater or equal $\mathbf{b}$, $\mathbf{a} \geq_{lex} \mathbf{b}$, if $\mathbf{a} >_{_{lex}} \mathbf{b}$ or $\mathbf{a} = \mathbf{b}$. Contrary to the standard vector inequality $\mathbf{a} \geqq \mathbf{b} \Leftrightarrow a_j \geq b_j \forall\ j$, the lexicographic order is complete which means that for any two vectors $\mathbf{a}$ and $\mathbf{b}$ either $\mathbf{a} \geq_{lex} \mathbf{b}$ or $\mathbf{b} \geq_{lex} \mathbf{a}$. Moreover, for any two different vectors $\mathbf{a} \neq \mathbf{b}$ either $\mathbf{a} >_{lex} \mathbf{b}$ or $\mathbf{b} >_{lex} \mathbf{a}$. Vector inequality $\mathbf{a} \geqq \mathbf{b}$ implies $\mathbf{a} \geq_{lex} \mathbf{b}$ but the

opposite implication is not valid. The lexicographic order is not continuous and it cannot be expressed in terms of any aggregation function. Nevertheless, it is a limiting case of the order (2.6) for the weighting aggregation functions $\theta(\mathbf{y}) = \sum_{j=1}^{m} w_j y_j$ defined by decreasing sequences of positive weights w_j with differences tending to the infinity.
The lexicographic order allows us to consider more complex solution concepts defined by several (say m) outcome functions $\theta_k : Y \rightarrow \Re$ to be maximized according to the lexicographic order. Thus one seeks a feasible solution $\mathbf{x}^0$ such that

$$(\theta_1(\mathbf{f}(\mathbf{x}^0)), \theta_2(\mathbf{f}(\mathbf{x}^0)), \ldots, \theta_m(\mathbf{f}(\mathbf{x}^0))) \geq_{lex} (\theta_1(\mathbf{f}(\mathbf{x})), \theta_2(\mathbf{f}(\mathbf{x})), \ldots, \theta_m(\mathbf{f}(\mathbf{x}))) \quad \forall \mathbf{x} \in Q.$$

In other words, the multiple criteria problem (2.1) is replaced with the lexicographic maximization problem

$$\operatorname{lex\,max} \{(\theta_1(\mathbf{f}(\mathbf{x})), \theta_2(\mathbf{f}(\mathbf{x})), \ldots, \theta_m(\mathbf{f}(\mathbf{x}))) : \mathbf{x} \in Q\}. \tag{2.7}$$

Problem (2.7) is not a standard mathematical programme. Nevertheless, the lexicographic inequality defines a linear order of vectors an therefore the lexicographic optimization is a well defined procedure where comparison of real numbers is replaced by lexicographic comparison of the corresponding vectors. In particular, the basic theory and algorithmic techniques for linear programming have been extended to the lexicographic case [10]. Certainly, the lexicographic optimization may also be treated as a sequential (hierarchical) optimization process where first $\theta_1(\mathbf{f}(\mathbf{x}))$ is maximized on the entire feasible set, next $\theta_2(\mathbf{f}(\mathbf{x}))$ is maximized on the optimal set, and so on. This may be implemented as in the following standard sequential algorithm.

Algorithm 1: Sequential algorithm for lexicographic maximization

Step 0: Put $k := 1$.

Step 1: Solve programme

$$P_k : \quad \max \{\tau_k;\ \tau_k \leq \theta_k(\mathbf{f}(\mathbf{x})),\ \tau_j^0 \leq \theta_j(\mathbf{f}(\mathbf{x}))\ \forall j < k,\ \mathbf{x} \in Q\}$$

and denote the resulting optimal solution by $(\mathbf{x}^0, \tau_k^0)$.

Step 2: If $k = m$ then **stop** ($\mathbf{x}^0$ is the optimal solution of problem (2.7)).
Otherwise, put $k := k + 1$ and go to **Step 1**.

Note that directly from the properties of the lexicographic order it follows that for any achievement functions θ_k the lexicographic optimization problem always has unique values of those functions, as stated in the following assertion.

Proposition 2 *For any two optimal solutions $\mathbf{x}^1, \mathbf{x}^2 \in Q$ of problem (2.7) the equalities $\theta_k(\mathbf{f}(\mathbf{x}^1)) = \theta_k(\mathbf{f}(\mathbf{x}^2))\ \forall\ k$ hold.*

The most commonly used lexicographic models are based on simple functions $\theta_j(\mathbf{y}) = y_j$ thus introducing an hierarchy of original outcomes. In such a case, according to Proposition 2 the optimal solution is unique in the criterion space.

Proposition 3 *In the case of problem (2.7) with $\theta_j(\mathbf{y}) = y_j \; \forall j \in M$, for any two optimal solutions $\mathbf{x}^1, \mathbf{x}^2 \in Q$ the equality $\mathbf{f}(\mathbf{x}^1) = \mathbf{f}(\mathbf{x}^2)$ holds and this unique outcome vector is nondominated.*

Applying to achievement vectors $\Theta(\mathbf{y})$ a linear cumulative map one gets the cumulated achievements

$$\bar{\theta}_k(\mathbf{y}) = \sum_{j=1}^{k} \theta_j(\mathbf{y}) \quad \text{for } k = 1, 2, \ldots, m. \tag{2.8}$$

Note that for any two vectors $\mathbf{y}', \mathbf{y}'' \in Y$ one gets

$$\Theta(\mathbf{y}') \geq_{lex} \Theta(\mathbf{y}'') \quad \Leftrightarrow \quad \bar{\Theta}(\mathbf{y}') \geq_{lex} \bar{\Theta}(\mathbf{y}''). \tag{2.9}$$

Hence, the following assertion is valid.

Proposition 4 *A feasible vector $\mathbf{x} \in Q$ is an optimal solution of problem (2.7), if and only if it is the optimal solution of the cumulated lexicographic problem*

$$\operatorname{lex\,max} \; \{(\bar{\theta}_1(\mathbf{f}(\mathbf{x})), \bar{\theta}_2(\mathbf{f}(\mathbf{x})), \ldots, \bar{\theta}_m(\mathbf{f}(\mathbf{x}))) : \mathbf{x} \in Q\}. \tag{2.10}$$

The lexicographic order may also be used to construct refinements of various solution concepts [23]. We focus on application of the lexicographic optimization to refine the Max-Min solution concept according to the Rawlsian theory of justice. Let $\langle \mathbf{a} \rangle = (a_{\langle 1 \rangle}, a_{\langle 2 \rangle}, \ldots, a_{\langle m \rangle})$ denote the vector obtained from $\mathbf{a}$ by rearranging its components in the non-decreasing order. That means $a_{\langle 1 \rangle} \leq a_{\langle 2 \rangle} \leq \ldots \leq a_{\langle m \rangle}$ and there exists a permutation π of set M such that $a_{\langle i \rangle} = a_{\pi(i)}$ for $j = 1, 2, \ldots, m$. Comparing lexicographically such ordered vectors $\langle \mathbf{y} \rangle$ one gets the so-called leximin order. The general problem considered in the balance of this paper depends on searching for the solutions that are maximal according to the leximin order. The problem called hereafter the Max-Min Fair (MMF) problem reads as follows.

P-MMF: Find $\mathbf{x}^0 \in Q$ such that $\langle \mathbf{f}(\mathbf{x}^0) \rangle \geq_{lex} \langle \mathbf{f}(\mathbf{x}) \rangle \; \forall \, \mathbf{x} \in Q$.

This problem may also be viewed as a standard lexicographic optimization (2.7) with the aggregation functions $\theta_j(\mathbf{y}) = y_{\langle j \rangle}$:

$$\operatorname{lex\,max} \; \{(\theta_1(\mathbf{f}(\mathbf{x})), \theta_2(\mathbf{f}(\mathbf{x})), \ldots, \theta_m(\mathbf{f}(\mathbf{x}))) : \mathbf{x} \in Q\}, \quad \text{where } \theta_j(\mathbf{y}) = y_{\langle j \rangle}. \tag{2.11}$$

Problem (2.11) represents the lexicographic Max-Min approach to the original multiple criteria problem (2.1). It is a refinement (regularization) of the standard Max-Min optimization, but this time, in addition to the smallest outcome, we also maximize the second smallest outcome (provided that the smallest one remains as large as possible), maximize the third smallest (provided that the two smallest remain as large as possible), and so on. Note that the lexicographic maximization is not applied to any specific order of the original criteria.

The lexicographic Max-Min is the only regularization approach of the Max-Min that satisfies the reduction (addition/deleting) principle [9]. Namely, if the individual outcome does not distinguish two solutions, then it does not affect the preference relation:

$$\langle (y'_1, \ldots, y'_j, y^*, y'_{j+1}, \ldots, y'_m) \rangle \geq_{lex} \langle (y''_1, \ldots, y''_j, y^*, y''_{j+1}, \ldots, y''_m) \rangle \quad \Leftrightarrow \quad \langle \mathbf{y}' \rangle \geq_{lex} \langle \mathbf{y}'' \rangle$$

For the lexicographic Max-Min one may also take advantage of Proposition 4. Applying the cumulative map (2.8) to ordered outcomes $\theta_i(\mathbf{y}) = y_{\langle i \rangle}$ one gets $\bar{\theta}_k(\mathbf{y}) = \sum_{i=1}^{k} \; y_{\langle i \rangle}$ expressing, respectively: the worst (smallest) outcome, the total of the two worst outcomes, the total of the three worst outcomes, etc. Following Proposition 4, solution of the **P-MMF** is equivalent to the lexicographic problem

$$\operatorname{lex\,max}\;\{(\bar{\theta}_1(\mathbf{y}), \bar{\theta}_2(\mathbf{y}), \ldots, \bar{\theta}_m(\mathbf{y})) : \mathbf{y} \leqq \mathbf{f}(\mathbf{x}),\;\; \mathbf{x} \in Q\}, \quad \text{where } \bar{\theta}_k(\mathbf{y}) = \sum_{j=1}^{k} \; y_{\langle j \rangle}. \tag{2.12}$$

Note that

$$\bar{\theta}_k(\mathbf{y}) = \sum_{j=1}^{k} \; y_{\langle j \rangle} = \min_{\pi \in \Pi} \sum_{j=1}^{k} \; y_{\pi(j)}$$

where the minimum is taken over all permutations of the index set M. Hence, $\bar{\theta}_k(\mathbf{y})$ is a concave piecewise linear function of $\mathbf{y}$ which, due to (2.12) guarantees several important properties of the lexicographic Max-Min solution itself.
Recall, that every optimal solution of the lexicographic Max-Min model is an efficient solution of the original multiple criteria optimization problem. Note that every lexicographic Max-Min solution is also an optimal solution of the standard Max-Min problem. Hence, by virtue of Proposition 1, the lexicographic Max-Min model, generates efficient solutions satisfying the perfect equity of individual outcomes, whenever such an efficient solution exists. When there does not exist any efficient solution with perfectly equal individual outcomes, then the lexicographic Max-Min model generates another efficient solution but, due to concave functions $\bar{\theta}_k(\mathbf{y})$, still providing equitability of individual outcomes with respect to the Pigou-Dalton principle of transfers [14]. The principle of transfers states, in the context considered here, that a transfer of small amount from an individual outcome to any relatively worse-off individual outcome results in a more preferred outcome vector. Indeed, the following assertion is valid.

Proposition 5 *For any outcome vector* $\mathbf{y} \in Y$

$$y_{j'} < y_{j''} \quad \Rightarrow \quad \langle \mathbf{y} + \varepsilon \mathbf{e}_{j'} - \varepsilon \mathbf{e}_{j''} \rangle >_{lex} \langle \mathbf{y} \rangle \quad \forall\; 0 < \varepsilon < y_{j''} - y_{j'} \tag{2.13}$$

where $\mathbf{e}_j$ *denotes the* j*-th unit vector.*

Proof. Let $\mathbf{y}^{\varepsilon} = \mathbf{y} + \varepsilon \mathbf{e}_{j'} - \varepsilon \mathbf{e}_{j''}$ for $\varepsilon < y_{j''} - y_{j'}$ and let $y_{\langle k' \rangle} = y_{j'}$, $y_{\langle k'' \rangle} = y_{j''}$. Then, $y_{j'} < y_{\langle k'' \rangle}$ and $\sum_{j=1}^{k} \; y^{\varepsilon}_{\langle j \rangle} \geq \sum_{j=1}^{k} \; y_{\langle j \rangle}$ for all $k = 1, 2, \ldots, m$ with at least one strict inequality for some $k' \leq k < k''$. Hence, $\langle \mathbf{y}^{\varepsilon} \rangle >_{lex} \langle \mathbf{y} \rangle$, due to (2.9). ■

Following Proposition 2, any two optimal solutions $\mathbf{x}^1, \mathbf{x}^2 \in Q$ of problem (2.11) result in the same ordered outcome vectors $\langle \mathbf{f}(\mathbf{x}^1) \rangle = \langle \mathbf{f}(\mathbf{x}^2) \rangle$. Hence, all the optimal solutions have the same distributions of outcomes. Nevertheless, they may generate different (differently ordered) outcome vectors themselves. The unique outcome vector is guaranteed, however, in the case of convex problems. It follows from the alternative convex formulation (2.12) of the MMF problem.

Proposition 6 *In the case of convex feasible set* Q *and concave objective functions* $f_j(\mathbf{x})$*, for any two optimal solutions* $\mathbf{x}^1, \mathbf{x}^2 \in Q$ *of problem **P-MMF** the equality* $\mathbf{f}(\mathbf{x}^1) = \mathbf{f}(\mathbf{x}^2)$ *holds.*

Proof. First of all, let us notice that problem **P-MMF** is equivalent (in the criterion space) to the following

$$\operatorname{lex}\max \{\langle \mathbf{y} \rangle : y_j \leq f_j(\mathbf{x}) \ \forall j, \ \mathbf{x} \in Q\} \tag{2.14}$$

and we need to prove that the problem has a unique optimal solution $\mathbf{y} \in Y$. Due to the convexity assumptions the attainable set Y is convex. Let, $\mathbf{y}^1 \neq \mathbf{y}^2 \in Y$ be optimal solutions of (2.14), thus $\langle \mathbf{y}^1 \rangle = \langle \mathbf{y}^2 \rangle$. Define $\mathbf{y}^\varepsilon = (1-\varepsilon)\mathbf{y}^1 + \varepsilon \mathbf{y}^2$ for some positive ε satisfying

$$0 < \varepsilon < \min_{y^1_{j'} \neq y^1_{j''}} |y^1_{j'} - y^1_{j''}| / \max_{y^1_{j'} \neq y^1_{j''}} |y^1_{j'} - y^1_{j''}|.$$

Due to the bound on ε, there exists a permutation π ordering both $\mathbf{y}^1$ and $\mathbf{y}^\varepsilon$, i.e., $y^1_{\pi(j)} \leq y^1_{\pi(j+1)}$ and $y^\varepsilon_{\pi(j)} \leq y^\varepsilon_{\pi(j+1)}$ for all $j = 1, \ldots, m-1$. Further, identifying the index j_o for which $y^1_{j_o}$ is the smallest value y^1_j such that $y^1_j \neq y^2_j$ one gets $y^\varepsilon_{\pi(j)} \geq y^1_{\pi(j)}$ for $j < j_o$ and $y^\varepsilon_{\pi(j_o)} > y^1_{\pi(j_o)}$ which contradicts optimality of $\mathbf{y}^1$. ■

The leximin order cannot be expressed in terms of any aggregation function. Nevertheless, it is a limiting case of the order (2.6) for the Ordered Weighted Aggregation functions $\theta(\mathbf{y}) = \sum_{j=1}^m w_j y_{\langle j \rangle}$ defined by decreasing sequences of positive weights w_j with differences tending to the infinity [36, 38].

3 Telecommunications network design examples

Below we shall give three examples showing how the MMF concept can be used in formulations of multi-commodity network flow problems related to telecommunications applications.

3.1 Routing design for networks with elastic traffic

The first example is a problem of finding flows in a network with given link capacities so as to obtain the MMF distribution of flow sizes. This type of problem is applicable to networks carrying the so-called elastic traffic, which means that traffic streams can adapt their intensity to the available capacity of the network [28].

Problem 1: *Routing optimization for MMF distribution of demand volumes*

indices

- $d = 1, 2, \ldots, D$ demands (pairs of nodes)
- $p = 1, 2, \ldots, P_d$ allowable paths for demand d
- $e = 1, 2, \ldots, E$ links

constants

- δ_{edp} equals 1 if link e belongs to path p of demand d; 0, otherwise
- c_e capacity of link e

variables

- x_{dp} flow (bandwidth) allocated to path p of demand d (non-negative continuous)
- X_d total flow (bandwidth) allocated to demand d (non-negative continuous), $\boldsymbol{X} = (X_1, X_2, \ldots, X_D)$

objective

$$\text{lex}\max\ (X_{\langle 1\rangle}, X_{\langle 2\rangle}, \ldots, X_{\langle D\rangle}) \tag{3.15a}$$

$$\tag{3.15b}$$

constraints

$$\textstyle\sum_p x_{dp} = X_d \qquad d = 1, 2, \ldots, D \tag{3.15c}$$

$$\textstyle\sum_d \sum_p \delta_{edp} x_{dp} \le c_e \qquad e = 1, 2, \ldots, E \tag{3.15d}$$

$$x_{dp} \ge 0 \qquad d = 1, 2, \ldots, D \quad p = 1, 2, \ldots, P_d \tag{3.15e}$$

In the above formulation, equation (3.15b) defines the total flow, X_d, allocated to demand d, and constraint (3.15c) assures that the link load (left-hand side) does not exceed the link capacity. A solution of Problem 1 for an example network is discussed in Appendix A.

3.2 Restoration design for networks with elastic traffic

The second example corresponds to the problem of designing an optimal strategy of elastic traffic flows restoration in case of network failures ([27, Chapter 13]). It is assumed that a set of network failure situations have been identified. The adopted failure model is such that a failure may reduce the capacity of one or more network links. The design should determine optimal capacities of links and for each failure situation the optimal size and routing of every traffic flow so as to obtain the MMF distribution of revenue for all network failure situations. It is assumed that the revenue generated by a single traffic flow is proportional to the logarithm of this flow's size.

Problem 2: *Flow restoration optimization for MMF distribution of revenues*

indices

$d = 1, 2, \ldots, D$	demands
$p = 1, 2, \ldots, P_d$	allowable paths for demand d
$e = 1, 2, \ldots, E$	links
$s = 1, 2, \ldots, S$	states (including normal operation state)

constants

δ_{ed}	equals 1 if link e belongs to the fixed path of demand d; 0, otherwise
r_{ds}	revenue from demand d in situation s
ξ_e	unit cost of link e
α_{es}	fractional availability coefficient of link e in situation s ($0 \le \alpha_{es} \le 1$)
B	assumed budget

variables

y_e	capacity of link e (non-negative continuous)
x_{dps}	flow allocated to path p of demand d in situation s (non-negative continuous)
X_{ds}	total flow allocated to demand d in situation s (non-negative continuous)
R_s	logarithmic revenue in situation s (continuous), $\boldsymbol{R} = (R_1, R_2, \ldots, R_S)$

objective

$$\text{lex}\max\ (R_{\langle 1\rangle}, R_{\langle 2\rangle}, \ldots, R_{\langle S\rangle}) \tag{3.16a}$$

constraints

$$X_{ds} = \sum_p x_{dps} \qquad d = 1, 2, \dots, D \quad s = 1, 2, \dots, S \tag{3.16b}$$
$$R_s = \sum_d r_{ds} \log X_{ds} \qquad s = 1, 2, \dots, S \tag{3.16c}$$
$$\sum_d \sum_p \delta_{edp} x_{dps} \leq \alpha_{es} y_e \qquad e = 1, 2, \dots, E \tag{3.16d}$$
$$\sum_e \xi_e y_e \leq B \tag{3.16e}$$
$$x_{dps} \geq 0 \qquad d = 1, 2, \dots, D \quad p = 1, 2, \dots, P_d \quad s = 1, 2, \dots, S \tag{3.16f}$$

3.3 Capacity protection design

The last example corresponds to the problem of designing the protection of network links' capacity [20]. It is assumed that the capacity of network links and the size and routing of all network flows are given. The design should determine how much capacity of each link should be freed and reserved so in case of any single-link failure the capacity of the failed link could be restored using the reserved protection capacity. In order to free the capacity of links the size of traffic flows should be reduced in such a way so as to obtain the MMF distribution of traffic flow sizes.

Problem 3: *Protection capacity optimization for MMF distribution of flow sizes*

indices

- $d = 1, 2, ..., D$ demands
- $p = 1, 2, ..., P_d$ allowable paths for demand d
- $e, \ell = 1, 2, ..., E$ links
- $q = 1, 2, ..., Q_\ell$ candidate restoration paths for link ℓ

constants

- h_d "reference" volume of demand d
- δ_{edp} equals 1 if link e belongs to path p realizing demand d; 0, otherwise
- c_e total capacity of link e
- $\beta_{\ell eq}$ equals 1 if link ℓ belongs to path q restoring link e; 0, otherwise

variables

- y_e resulting normal capacity of link e
- x_{dp} normal flow realizing demand d on path p
- w_e protection capacity of link e
- z_{eq} flow restoring capacity of link e on path q
- X_d normalized realized demand volume for demand d, $\boldsymbol{X} = (X_1, X_2, \dots, X_D)$

objective

$$\text{lex max } (X_{\langle 1 \rangle}, X_{\langle 2 \rangle}, \dots, X_{\langle D \rangle}) \tag{3.17a}$$

constraints

$$X_d = \sum_p x_{dp}/h_d \quad d = 1, 2, ..., D \tag{3.17b}$$
$$w_e + u_e \leq c_e \quad e = 1, 2, ..., E \tag{3.17c}$$
$$\sum_d \sum_p \delta_{edp} x_{dp} \leq y_e, \quad e = 1, 2, ..., E \tag{3.17d}$$
$$y_e \leq \sum_q z_{eq} \quad e = 1, 2, ..., E \tag{3.17e}$$
$$\sum_q \beta_{\ell eq} z_{eq} \leq w_\ell \quad \ell = 1, 2, ..., E \quad e = 1, 2, ..., E \quad \ell \neq e \tag{3.17f}$$
$$x_{dp} \geq 0 \qquad d = 1, 2, \ldots, D \quad p = 1, 2, \ldots, P_d \tag{3.17g}$$

Note that the lexicographic Max-Min solution assures that all demand volumes will be in the worst case decreased by the same optimal proportion r^*, since in the optimal solution $\sum_p x^*_{dp} \geq r^* h_d$, $d = 1, 2, ..., D$, for some number r^*, such that $\sum_p x^*_{dp} = r^* h_d$ for some d.

3.4 Non-convex extensions of the example problems

All three problems presented in the previous subsections have convex sets of feasible solutions. As we will see in Section 4, this property allows for efficient solution algorithms of the introduced problems, but, unfortunately, it is not always present in telecommunications problems. For instance, we may require that the demand volumes are realized only on single paths and that the choice of these single paths is subject to optimization. This requirement usually leads to Mixed-Integer Programme (MIP) formulations. In particular, Problem 1 in the single-path version requires additional multiple choice constraints to enforce nonbifurcated flows. Assuming existence of some constants U_d upper bounding the largest possible total flows X_d, this can be implemented with additional binary (flow assignement) variables u_{dp} used to limit the number of positive flows x_{dp} with constraints:

$$x_{dp} \leq U_d u_{dp} \qquad d = 1, 2, \ldots, D \quad p = 1, 2, \ldots, P_d \tag{3.18a}$$
$$\sum_p u_{dp} = 1 \qquad d = 1, 2, \ldots, D \tag{3.18b}$$
$$u_{dp} \in \{0, 1\} \qquad d = 1, 2, \ldots, D \quad p = 1, 2, \ldots, P_d \tag{3.18c}$$

In fact, as demonstrated in [13], such a modification makes Problem 1 *NP*-complete. The same requirement can be introduced to Problems 2 and 3 as well.
Another requirement leading to non-convex MIP problems is the modularity of the link capacity, which means that link capacities should be multiples of a given module C. Then, capacity variables become non-negative integers and respective constraints change. For example, for Problem 2 variables y_e are non-negative integers and constraints (3.16d) take the form

$$\sum_d \sum_p \delta_{edp} x_{dps} \leq \alpha_{es} C y_e, \qquad e = 1, 2, \ldots, E. \tag{3.19}$$

Certainly, the capacity variables in Problem 3 can also be made integral.

4 MMF solution algorithms

4.1 Sequential Max-Min algorithms for convex problems

The (point-wise) ordering of outcomes causes that the lexicographic Max-Min problem (2.11) is, in general, hard to implement. Note that the quantity $y_{\langle 1 \rangle}$ representing the worst outcome can be easily computed directly by the maximization:

$$y_{\langle 1\rangle} = \max \ r_1 \quad \text{subject to} \quad r_1 \leq y_j \quad \text{for } j = 1, 2, \dots, m.$$

Similar simple formula does not exist for the further ordered outcomes $y_{\langle k\rangle}$. Nevertheless, for convex problems it is possible to use iterative algorithms for finding the consecutive values of the (unknown) optimal unique vector $\mathbf{T}^0 = (T_1^0, T_2^0, \dots, T_m^0) = \langle \mathbf{f}(\mathbf{x}^0)\rangle$ by solving a sequence of properly defined Max-Min problems. Such algorithms are described below. Suppose B is a subset of the index set M, $B \subseteq M$, and let $\mathbf{t}^B = (t_j : \ j \in B)$ be a $|B|$-vector. Also, let B' denote the set complementary to B : $B' = M \backslash B$. For given B and $\mathbf{t}^B$ we define the following convex mathematical programming problem in variables $\mathbf{x}$ and τ:
$\mathbf{P}(B,\mathbf{t}^B)$:

$$\begin{aligned}
&\textbf{maximize} && \tau && && \text{(4.20a)}\\
&\textbf{subject to} && f_j(\mathbf{x}) \geq \tau && j \in B' && \text{(4.20b)}\\
& && f_j(\mathbf{x}) \geq t_j^B && j \in B && \text{(4.20c)}\\
& && \mathbf{x} \in \mathbf{X}. && && \text{(4.20d)}
\end{aligned}$$

It is clear that the solution τ^0 of the convex problem $\mathbf{P}(\emptyset,\emptyset)$ (defined by (4.20) for empty set B and empty sequence $\mathbf{t}^B$) will yield the smallest value of $\mathbf{T}^0$, i.e. the value T_1^0 (and possibly some other consecutive entries of $\mathbf{T}^0$). This observation suggests the following algorithm for solving problem **P-MMF** specified by (2.11).

Algorithm 2: Straightforward algorithm for solving Problem P-MMF

Step 0: Put $B := \emptyset$ (empty set) and $\mathbf{t}^B := \emptyset$ (empty sequence).

Step 1: If $B = M$ then **stop** ($\langle \mathbf{t}^B\rangle$ is the optimal solution of problem **P-MMF**, i.e. $\langle \mathbf{t}^B\rangle = \mathbf{T}^0$). Else, solve programme $\mathbf{P}(B, \mathbf{t}^B)$ and denote the resulting optimal solution by $(\mathbf{x}^0, \tau^0)$.

Step 2: For each index $k \in B'$ such that $f_k(\mathbf{x}^0) = \tau^0$ solve the following test problem $\mathbf{T}(B, \mathbf{t}^B, \tau^0, k)$:

$$\begin{aligned}
&\textbf{maximize} && f_k(\mathbf{x}) && && \text{(4.21a)}\\
&\textbf{subject to} && f_j(\mathbf{x}) \geq \tau^0 && j \in B' \backslash \{k\} && \text{(4.21b)}\\
& && f_j(\mathbf{x}) \geq t_j^B && j \in B && \text{(4.21c)}\\
& && \mathbf{x} \in \mathbf{X}. && && \text{(4.21d)}
\end{aligned}$$

If for optimal $\mathbf{x}^1$, while solving test $\mathbf{T}(B, \mathbf{t}^B, \tau^0, k)$ we have $f_k(\mathbf{x}^1) = \tau^0$, then we put $B := B \cup \{k\}$ and $t_k := \tau^0$.

Step 3: Go to **Step 1**.

It can happen that as a result of solving the test in Step 2 for some index $k \in B'$, it will turn out that $f_l(\mathbf{x}^1) > \tau^0$ for some other, not yet tested, index $l \in B'$ $(l \neq k)$. In

such an (advantageous) case, the objective function with index l does not have to be tested, as its value can be further increased without disturbing the maximal values $\mathbf{t}^B$. Observe that set B is the current set of blocking indices, i.e. the indices j for which the value $f_j(\mathbf{x}^0)$ is equal to t_j^B in every optimal solution of Problem **P-MMF**. Note also, that although the tests in Step 2 are performed separately for individual indices $j \in B'$, the values of objective functions f_j for the indices $j \in B'$, where set B' is results from Step 2, can be simultaneously increased above the value of τ^0 in the next execution of Step 1. This follows from convexity of the set defined by constraints (4.21b-d): if $f_j(\mathbf{x}^j) = a^j > \tau^0$ and $\mathbf{x}^j$ satisfies (4.21b-d), then a convex combination of the points $\mathbf{x}^j$, $\mathbf{x} = \sum_{j \in B'} \alpha^j \mathbf{x}^j$ ($\sum_{j \in B'} \alpha^j = 1,\ \alpha^j > 0,\ j \in B'$) also satisfies (4.21b-d), and $f_j(\mathbf{x}) > \tau^0$ for all $j \in B'$.
Another version of Algorithm 2 may be more efficient, provided that the complexity of problems (4.20) and (4.21) is similar.

Algorithm 3: Algorithm for solving Problem P-MMF

Step 0:	Put $B := \emptyset$ and $\mathbf{t}^B := \emptyset$.
Step 1:	If $B = M$ then **stop** ($\langle \mathbf{t}^B \rangle$ is the optimal solution of problem **P-MMF**, i.e. $\langle \mathbf{t}^B \rangle = \mathbf{T}^0$). Else, solve programme $\mathbf{P}(B, \mathbf{t}^B)$ and denote the resulting optimal solution by $(\mathbf{x}^0, \tau^0)$.
Step 2:	Start solving the test problem $\mathbf{T}(B, \mathbf{t}^B, \tau^0, k)$ for all indices $k \in B'$ such that $f_k(\mathbf{x}^0) = \tau^0$. When the first $k \in B'$ with $f_k(\mathbf{x}^1) = \tau^0$ is detected, then put $B := B \cup \{k\}$ and $t_k := \tau^0$, and go to **Step 3**.
Step 3:	Go to **Step 1**.

The idea behind the modification in Algorithm 3 is that in total it may involve solving less instances of problems $\mathbf{P}(B, \mathbf{t}^B)$ and $\mathbf{T}(B, \mathbf{t}^B, \tau^0, k)$ than Algorithm 2. If at optimum $\mathbf{x}^0$ all values $f_j(\mathbf{x}^0)$ are the same (equal to 0), then Algorithm 2 will require solving $m+1$ problems (problem $\mathbf{P}(\emptyset, \emptyset)$ and m tests $\mathbf{T}(\emptyset, \emptyset, \tau^0, k)$ for $k = 1, 2, \ldots, m$), whilst Algorithm 3 will require solving $2m+1$ problems (problem $\mathbf{P}(\emptyset, \emptyset)$, m tests $\mathbf{T}(B, \mathbf{t}^B, \tau^0, k)$ and m problems $\mathbf{P}(B, \mathbf{t}^B)$). Hence, in this case, Algorithm 3 requires solving $O(m)$ more problems than Algorithm 2. Now let us consider a somewhat opposite case where all values $f_j(\mathbf{x}^0)$ are different. Additionally, assume that all optimal solutions $\mathbf{x}$ of the consecutively solved problems $\mathbf{P}(B, \mathbf{t}^B)$ and $\mathbf{T}(B, \mathbf{t}^B, \tau^0, k)$ yield the same values $f_j(\mathbf{x})$ for $j \in B'$. In this case Algorithm 3 will require solving $O(m^2/4)$ problems, while Algorithm 2 – $O(m^2/2)$ problems. This means that Algorithm 2 requires solving $O(m^2/4)$ more problems than Algorithm 3; this is a substantial difference.
Both algorithms presented above can be time consuming due to excessive number of problems $\mathbf{P}(B, \mathbf{t}^B)$ and $\mathbf{T}(B, \mathbf{t}^B, \tau^0, k)$ that may have to be solved in the iteration process. Therefore, below we give an alternative algorithm which is very fast provided that dual optimal variables problems $\mathbf{P}(B, \mathbf{t}^B)$ can be effectively computed (this is for instance the case for linear programmes and the simplex algorithm).
Suppose $\boldsymbol{\lambda} = (\lambda_j)_{j \in B'}$ denotes the vector of dual variables (multipliers) associated with

constraints (4.20b). It leads to the following Lagrangian function for problem $\mathbf{P}(B, \mathbf{t}^B)$:

$$L(\mathbf{x};\tau;\boldsymbol{\lambda}) = -\tau + \sum\nolimits_{j\in B'} \lambda_j(\tau - f_j(\mathbf{x})) = (\sum\nolimits_{j\in B'} \lambda_j - 1)\tau - \sum\nolimits_{j\in B'} \lambda_j f_j(\mathbf{x}). \quad (4.22)$$

The domain of Lagrangian (4.22) is defined by

$$\mathbf{x} \in \mathbf{Y} \text{ (where } \mathbf{Y} \text{ is determined by constraints (4.20c} - \text{d))} \quad (4.23a)$$
$$-\infty < \tau < +\infty \quad (4.23b)$$
$$\boldsymbol{\lambda} \geq \mathbf{0}. \quad (4.23c)$$

Hence, the dual function is formally defined as

$$W(\boldsymbol{\lambda}) = \min\nolimits_{\tau,\mathbf{x}\in\mathbf{Y}} L(\mathbf{x},\tau;\boldsymbol{\lambda}) \quad \boldsymbol{\lambda} \geq \mathbf{0} \quad (4.24)$$

and the dual problem reads:

$$\textbf{maximize } W(\boldsymbol{\lambda}) \textbf{ over } \boldsymbol{\lambda} \geq \mathbf{0}. \quad (4.25)$$

The following proposition can be proved [27].

Proposition 7 *Let $\boldsymbol{\lambda}^0$ be the vector of optimal dual variables solving the dual problem (4.25). Then*

(1)

$$\sum\nolimits_{j\in B'} \lambda_j^0 = 1 \quad (4.26)$$

(2) if $\lambda_j^0 > 0$ for some $j \in B'$, then $f_j(\mathbf{x})$ cannot be improved, i.e. $f_j(\mathbf{x}^0) = \tau^0$ for every optimal primal solution $(\mathbf{x}^0, \tau^0)$ of (4.20).

Note that in general the inverse of (2) in Proposition 7 does not hold: $\lambda_j^0 = 0$ does not necessarily imply that $f_j(\mathbf{x})$ can be improved (for an example see [27, 28]). In fact, it can be proved [27, Chpt. 13] that the inverse implication holds if and only if set B is regular (set B is called regular if for any non-empty proper subset G of B, in the modified formulation $\mathbf{P}(B\backslash G, \mathbf{t}^{B\backslash G})$ the value of $f_k(\mathbf{x})$ can be improved for at least one of the indices $k \in B\backslash G$).
Whether or not the consecutive sets B are regular, the following algorithm solves problem **P-MMF**.

Algorithm 4: Algorithm for solving Problem P-MMF based on dual variables

Step 0: Put $B := \emptyset$ and $\mathbf{t}^B := \emptyset$.

Step 1: If $B = M$ then **stop** ($\langle \mathbf{t}^B \rangle$ is the optimal solution of problem **P-MMF**, i.e. $\langle \mathbf{t}^B \rangle = \mathbf{T}^0$). Else, solve programme $\mathbf{P}(B, \mathbf{t}^B)$ and denote the resulting optimal solution by $(\mathbf{x}^0, \tau^0; \boldsymbol{\lambda}^0)$.

Step 2: Put $B := B \cup \{j \in B' : \lambda_j^0 > 0\}$.

Step 3: Go to **Step 1**.

Observe that if for some $j \in B'$ with $\lambda_j^0 = 0$, $f_j(\mathbf{x})$ cannot be further improved, then in Step 1 the value of τ^0 will not be improved; still at least one such index j will be detected (due to property (3.5)) and included into set B in the next execution of Step 2. The regularity of set B simply ensures that in each iteration at least one $f_j(\mathbf{x})$ ($j \in B'$) will be improved.
In the case of LP problems, the dual quantities used in Algorithm 4 can be obtained directly from the simplex tableau. Indeed, it was a basis of early implementations of the lexicographic Max-Min solution for LP problems [1, 2, 12].

4.2 Conditional means

The sequential Max-Min algorithms can be applied only to convex problems, because, in general, it is likely that there does not exist a blocking index set B allowing for iterative processing. This can be illustrated with the following small example. Problem

$$\operatorname{lex\,max} \{\langle(x_1 + 2x_2, 3x_1 + x_2)\rangle : x_1 + x_2 = 1,\ x_1, x_2 \in \{0,1\}\}$$

has two feasible vectors $\mathbf{x}^1 = (1,0)$, $\mathbf{x}^2 = (0,1)$ and corresponding outcomes $\mathbf{y}^1 = (1,3)$, $\mathbf{y}^2 = (2,1)$. Obviously, $\mathbf{x}^1$ is the MMF optimal solution as $\langle(1,3)\rangle >_{lex} \langle(2,1)\rangle$. One can easily verify that both feasible solutions are optimal for Max-Min problem

$$\max \{\min\{x_1 + 2x_2, 3x_1 + x_2\} : x_1 + x_2 = 1,\ x_1, x_2 \in \{0,1\}\}$$

but neither f_1 nor f_2 is a blocking outcome allowing to define the second level Max-Min optimization problem to maximize the second worst outcome. For the same reason, the sequential algorithm may fail for the single-path version of the routing optimization for the MMF distribution of demand volumes and other discrete models (refer to Section 3.4). Following Yager [37], a direct, although requiring the use of integer variables, formula can be given for any $y_{\langle k\rangle}$. Namely, for any $k = 1, 2, \ldots, m$ the following formula is valid:

$$\begin{array}{rcll} y_{\langle k\rangle} & = & \max\ r_k & \\ & \text{s.t.} & & \\ & & r_k - y_j \leq C z_{kj},\ z_{kj} \in \{0,1\} & \text{for } j = 1, 2, \ldots, m \\ & & \displaystyle\sum_{j=1}^{m} z_{kj} \leq k - 1 & \end{array} \tag{4.27}$$

where C is a sufficiently large constant (larger than any possible difference between various individual outcomes y_j) which allows us to enforce inequality $r_k \leq y_j$ for $z_{kj} = 0$ while ignoring it for $z_{kj} = 1$. Note that for $k = 1$ all binary variables z_{1j} are forced to 0 thus reducing the optimization in this case to the standard LP model. However, for any other $k > 1$ all m binary variables z_{kj} are an important part of the model. Nevertheless, with the use of auxiliary integer variables, any MMF problem (either convex or non-convex) can be formulated as the standard lexicographic maximization with directly defined achievement functions

$$\begin{aligned}
\operatorname{lex\,max} \quad & (r_1, r_2, \ldots, r_m) && (4.28a)\\
\text{s.t.} \quad & \\
& \mathbf{x} \in Q && (4.28b)\\
& r_k - f_j(\mathbf{x}) \leq C z_{kj}, \;\; z_{kj} \in \{0,1\} \quad \text{for } j,k = 1,2,\ldots,m && (4.28c)\\
& \sum_{j=1}^{m} z_{kj} \leq k-1 \quad \text{for } k = 1,2,\ldots,m. && (4.28d)
\end{aligned}$$

Recall that one may take advantage of the formulation (2.12) with cumulated criteria $\bar{\theta}_k(\mathbf{y}) = \sum_{i=1}^{k} y_{\langle i \rangle}$ expressing, respectively: the worst (smallest) outcome, the total of the two worst outcomes, the total of the three worst outcomes, etc. When normalized by k the quantities $\mu_k(\mathbf{y}) = \bar{\theta}_k(\mathbf{y})/k$ can be interpreted as the worst conditional means [24]. The optimization formula (4.27) for $y_{\langle k \rangle}$ can easily be extended to define $\bar{\theta}_k(\mathbf{y})$. Namely, for any $k = 1,2,\ldots,m$ the following formula is valid:

$$\begin{aligned}
\bar{\theta}_k(\mathbf{y}) = \quad & \max \;\; k r_k - \sum_{j=1}^{m} d_{kj} \\
\text{s.t.} \quad & \\
& r_k - y_j \leq d_{kj}, \;\; d_{kj} \geq 0 \qquad \text{for } j = 1,2,\ldots,m \\
& d_{kj} \leq C z_{kj}, \;\; z_{kj} \in \{0,1\} \qquad \text{for } j = 1,2,\ldots,m \\
& \sum_{j=1}^{m} z_{kj} \leq k-1
\end{aligned} \tag{4.29}$$

where C is a sufficiently large constant. However, the optimization problem defining the cumulated ordered outcome can be dramatically simplified since all its binary variables (and the related constraints) turn out to be redundant. First let us notice that for any given vector $\mathbf{y} \in \Re^m$, the cumulated ordered value $\bar{\theta}_k(\mathbf{y})$ can be found as the optimal value of the following LP problem:

$$\begin{aligned}
\bar{\theta}_k(\mathbf{y}) \quad = \quad & \min \sum_{j=1}^{m} y_j u_{kj} \\
\text{s.t.} \quad & \\
& \textstyle\sum_{j=1}^{m} u_{kj} = k, \;\; 0 \leq u_{kj} \leq 1 \quad \text{for } j = 1,2,\ldots,m.
\end{aligned} \tag{4.30}$$

The above problem is an LP for a given outcome vector $\mathbf{y}$ while it becomes nonlinear for $\mathbf{y}$ being a variable. This difficulty can be overcome by taking advantage of the LP dual to (4.30) as shown in the following assertion.

Proposition 8 *For any given vector $\mathbf{y} \in \Re^m$, the cumulated ordered coefficient $\bar{\theta}_k(\mathbf{y})$ can be found as the optimal value of the following LP problem:*

$$\begin{aligned}
\bar{\theta}_k(\mathbf{y}) \quad = \quad & \max \;\; k r_k - \sum_{j=1}^{m} d_{kj} \\
\textit{s.t.} \quad & \\
& r_k - y_j \leq d_{kj}, \;\; d_{kj} \geq 0 \quad \textit{for } j = 1,2,\ldots,m.
\end{aligned} \tag{4.31}$$

Proof. In order to prove the proposition it is enough to notice that problem (4.31) is the LP dual of problem (4.30) with variable r_k corresponding to the equation $\sum_{j=1}^{m} u_{kj} = k$ and variables d_{kj} corresponding to upper bounds on u_{kj}. ■

It follows from Proposition 8 that

$$\bar{\theta}_k(\mathbf{f}(\mathbf{x})) = \max \{kr_k - \sum_{j=1}^{m} d_{kj} \;:\; \mathbf{x} \in Q; \quad r_k - f_j(\mathbf{x}) \leq d_{kj},\; d_{kj} \geq 0 \quad \text{for } j \in M\}$$

or in a more compact form $\bar{\theta}_k(\mathbf{f}(\mathbf{x})) = \max \{kr_k - \sum_{j=1}^{m} (f_j(\mathbf{x}) - r_k)_+ : \mathbf{x} \in Q\}$ where $(.)_+$ denotes the nonnegative part of a number and r_k is an auxiliary (unbounded) variable. The latter, with the necessary adaptation to the minimized outcomes in location problems, is equivalent to the computational formulation of the k–centrum model introduced in [26]. Hence, Proposition 8 provides an alternative proof of that formulation.
Due to Proposition 4, the lexicographic Max-Min problem (2.11) is equivalent to the lexicographic maximization of conditional means

$$\operatorname{lex\,max} \{(\mu_1(\mathbf{f}(\mathbf{x})), \mu_2(\mathbf{f}(\mathbf{x})), \ldots, \mu_m(\mathbf{f}(\mathbf{x}))) \;:\; \mathbf{x} \in Q\}.$$

Following Proposition 8, the above leads us to a standard lexicographic optimization problem with predefined linear criteria:

$$\operatorname{lex\,max} \quad (r_1 - \sum_{j=1}^{m} d_{1j},\; r_2 - \frac{1}{2}\sum_{j=1}^{m} d_{2j}, \ldots, r_m - \frac{1}{m}\sum_{j=1}^{m} d_{mj}) \tag{4.32a}$$

s.t.

$$\mathbf{x} \in Q \tag{4.32b}$$

$$d_{kj} \geq r_k - f_j(\mathbf{x}) \quad \text{for } j,k = 1,2,\ldots,m \tag{4.32c}$$

$$d_{kj} \geq 0 \quad \text{for } j,k = 1,2,\ldots,m. \tag{4.32d}$$

Note that this direct lexicographic formulation remains valid for nonconvex (e.g. discrete) feasible sets Q, where the standard sequential approaches [16, 17] are not applicable [21]. Model (4.32) preserves the problem convexity when the original problem is defined with convex feasible set Q and concave objective functions f_j. In particular, for an LP original problem it remains within the LP class while introducing $m^2 + m$ auxiliary variables and m^2 constraints. Thus, for many problems with not too large number of criteria m, problem (4.32) can easily be solved directly. Although, in general, for convex problems such an approach seems to be less efficient than the sequential algorithms discussed in the previous subsection. The latter may require m iterative steps only in the worst case (only one blocking variable at each step), while typically there are more than two blocking variables identified at each step which reduces significantly the number of steps. The direct model (4.32) essentially requires the sequential lexicographic Algorithm 1 with m steps.
Further research on the increase of computational efficiency of model (4.32) seems to be very promising. Note that all lexicographic criteria of this problem express the conditional means which are monotonic with respect to increasing k. While solving the lexicographic problem with the standard sequential Algorithm 1, one needs to solve at Step 2 the

following maximization problem:

$$\max\ \{\tau_k\ :\ \tau_k \le r - w\sum_{j=1}^{m} d_j;\ \mu_l(\mathbf{f}(\mathbf{x})) \ge \tau_l^0\ \forall l < k;\ \mathbf{x} \in Q;\ r - f_j(\mathbf{x}) \le d_j,\ d_j \ge 0 \forall j\}$$

where $w = 1/k$. It may occur that the optimal solution of the above problem remains also optimal for smaller coefficients $w = 1/\kappa$ thus defining conditional means for $\kappa > k$. In such a case, one may advance the iterative process to $\kappa + 1$ instead of $k + 1$. Hence, some parametric optimization techniques may allow us to reduce the number of iterations to the same level as in the sequential Max-Min algorithms.

Note that model (4.32) offers also a possibility to build some approximations to the strict MMF solution as it allows us to build lexicographic problems taking into account only a selected grid of indices k. In particular, the so-called augmented Max-Min solution concept, commonly used in the multiple criteria optimization [22, 35], is such an approximation, although very rough as based only on μ_1 and μ_k

$$\mathrm{lex}\max\{(r_1,\ \frac{1}{m}\sum_{j=1}^{m} f_j(\mathbf{x})) : r_1 \le f_j(\mathbf{x}) \quad \text{for } j = 1, 2, \ldots, m,\ \mathbf{x} \in Q\}.$$

4.3 Distribution approach

For some specific classes of discrete, or rather combinatorial, optimization problems, one may take advantage of the finiteness of the set of all possible values of functions f_j on the finite set of feasible solutions. The ordered outcome vectors may be treated as describing a distribution of outcomes generated by a given decision $\mathbf{x}$. In the case when there exists a finite set of all possible outcomes of the individual objective functions, we can directly describe the distribution of outcomes with frequencies of outcomes. Let $V = \{v_1, v_2, \ldots, v_r\}$ (where $v_1 < v_2 < \cdots < v_r$) denote the set of all attainable outcomes (all possible values of the individual objective functions f_j for $\mathbf{x} \in Q$). We introduce integer functions $h_k(\mathbf{y})$ ($k = 1, 2, \ldots, r$) expressing the number of values v_k in the outcome vector $\mathbf{y}$. Having defined functions h_k we can introduce cumulative distribution functions:

$$\bar{h}_k(\mathbf{y}) = \sum_{l=1}^{k} h_l(\mathbf{y})\ , \quad \text{for } k = 1, 2, \ldots, r. \tag{4.33}$$

Function $\bar{h}_k$ expresses the number of outcomes smaller or equal to v_k. Since we want to maximize all the outcomes, we are interested in the minimization of all functions $\bar{h}_k$. Indeed, the following assertion is valid [22]. For outcome vectors $\mathbf{y}', \mathbf{y}'' \in V^m$, $\langle \mathbf{y}' \rangle \ge \langle \mathbf{y}'' \rangle$ if and only if $\bar{h}_k(\mathbf{y}') \le \bar{h}_k(\mathbf{y}'')$ for all $k = 1, 2, \ldots, r$. This equivalence allows to express the lexicographic Max-Min solution concept for problem (2.1) in terms of the standard lexicographic minimization problem with objectives $\bar{\mathbf{h}}(\mathbf{f}(\mathbf{x}))$:

$$\mathrm{lex}\min\ \{(\bar{h}_1(\mathbf{f}(\mathbf{x})), \bar{h}_2(\mathbf{f}(\mathbf{x})), \ldots, \bar{h}_r(\mathbf{f}(\mathbf{x})))\ :\ \mathbf{x} \in Q\}. \tag{4.34}$$

Proposition 9 *A feasible solution $\mathbf{x} \in Q$ is an optimal solution of the* ***P-MMF*** *problem, if and only if it is an optimal solution of the lexicographic problem (4.34).*

The quantity $\bar{h}_k(\mathbf{y})$ can be computed directly by the minimization:

$$
\begin{aligned}
\bar{h}_k(\mathbf{y}) \quad &= \quad \min \sum_{j=1}^{m} z_{kj} \\
\text{s.t.} \quad & \\
& v_{k+1} - y_j \leq C z_{kj}, \;\; z_{kj} \in \{0,1\} \quad \text{for } j = 1,2,\ldots,m,
\end{aligned}
$$

where C is a sufficiently large constant. Note that $\bar{h}_r(\mathbf{y}) = m$ for any $\mathbf{y}$ which means that the r-th criterion is always constant and therefore redundant in (4.34). Hence, the lexicographic problem (4.34) can be formulated as the following mixed integer problem:

$$
\begin{aligned}
\operatorname{lex\,min} \quad & \left[\sum_{j=1}^{m} z_{1j}, \; \sum_{j=1}^{m} z_{2j}, \ldots, \; \sum_{j=1}^{m} z_{r-1,j} \right] \\
\text{s.t.} \quad & \\
& v_{k+1} - f_j(\mathbf{x}) \leq C z_{kj} \quad \text{for } j = 1,2,\ldots,m, \quad k = 1,2,\ldots,r-1, \\
& z_{kj} \in \{0,1\} \quad \text{for } j = 1,2,\ldots,m, \quad k = 1,2,\ldots,r-1, \\
& \mathbf{x} \in Q.
\end{aligned}
\tag{4.35}
$$

Krarup and Pruzan [15] have shown that, in the case of discrete location problems, the use of the minisum solution concept with the outcomes raised to a sufficiently large power is equivalent to the use of the minimax solution concept. Formulation (4.34) allows us to extend such an approach to the lexicographic Max-Min solution concept. Note that the achievements functions in (4.34) can be rescaled with corresponding values $v_{k+1} - v_k$. When the differences among outcome values are large enough then the lexicographic minimization corresponds to the one-level optimization of the total of achievements which is equivalent to minimization of the sum of the original outcomes. In general, as shown by Burkard and Rendl [4], there is a possibility to replace then the lexicographic Max-Min objective function with an equivalent linear function on rescaled outcomes. Algorithms developed in [4, 5] take advantage of finiteness of the set of outcome values and they depend on making (explicitly or implicitly) differences among the outcomes larger (without changing their order) which does not affect the lexicographically maximal solutions of problem (2.11). When the differences are large enough the optimal solution of the maxisum problem is also the lexicographic Max-Min solution. In general, an unrealistically complicated scaling function may be necessary to generate large enough differences among different but very close outcomes. Therefore, the outcomes should be mapped first on the set of integer variables (numbered) to normalize the minimum difference, like in [4, 5] approaches. All these transformations are eligible in the case of finite outcome set. Nevertheless, while solving practical problems, large differences among coefficients may cause serious computational problems. Therefore, such approaches are less useful for large scale problems typically arriving in telecommunications network design.
Taking advantage of possible weighting and cumulating achievements in lexicographic optimization, one may eliminate auxiliary integer variables from the achievement functions. For this purpose we weight and cumulate vector $\bar{\mathbf{h}}(\mathbf{y})$ to get:

$$
\hat{h}_1(\mathbf{y}) = 0 \quad \text{and} \quad \hat{h}_k(\mathbf{y}) = \sum_{l=1}^{k-1} (v_{l+1} - v_l)\bar{h}_l(\mathbf{y}) \quad \text{for } k = 2,\ldots,r. \tag{4.36}
$$

Due to Proposition 4 and positive differences $v_{l+1} - v_l > 0$, the lexicographic minimization problem (4.34) is equivalent to the lexicographic problem with objectives $\hat{\mathbf{h}}(\mathbf{f}(\mathbf{x}))$:

$$\operatorname{lex\,min}\ \{(\hat{h}_1(\mathbf{f}(\mathbf{x})), \hat{h}_2(\mathbf{f}(\mathbf{x})), \ldots, \hat{h}_r(\mathbf{f}(\mathbf{x})))\ :\ \mathbf{x} \in Q\} \tag{4.37}$$

which leads us to the following assertion.

Proposition 10 *A feasible solution $\mathbf{x} \in Q$ is an optimal solution of the **P-MMF** problem, if and only if it is an optimal solution of the lexicographic problem (4.37).*

Actually, vector function $\hat{\mathbf{h}}(\mathbf{y})$ provides a unique description of the distribution of coefficients of vector $\mathbf{y}$, i.e., for any $\mathbf{y}', \mathbf{y}'' \in V^m$ one gets: $\hat{\mathbf{h}}(\mathbf{y}') = \hat{\mathbf{h}}(\mathbf{y}'') \quad \Leftrightarrow \quad \langle \mathbf{y}' \rangle = \langle \mathbf{y}'' \rangle$. Moreover, $\hat{\mathbf{h}}(\mathbf{y}') \leq \hat{\mathbf{h}}(\mathbf{y}'')$ if and only if $\bar{\Theta}(\mathbf{y}') \geq \bar{\Theta}(\mathbf{y}'')$ [22].
Note that $\hat{h}_1(\mathbf{y}) = 0$ for any $\mathbf{y}$ which means that the first criterion is constant and redundant in problem (4.37). Moreover, putting (4.33) into (4.36) allows us to express all achievement functions $\hat{h}_k(\mathbf{y})$ as a piece wise linear functions of $\mathbf{y}$:

$$\hat{h}_k(\mathbf{y}) = \sum_{j=1}^{m} (v_k - y_j)_+ = \sum_{j=1}^{m} \max\{v_k - y_j, 0\} \quad \text{for } k = 1, 2, \ldots, r. \tag{4.38}$$

Hence, the quantity $\hat{h}_k(\mathbf{y})$ can be computed directly by the following minimization:

$$\begin{aligned} \hat{h}_k(\mathbf{y}) \quad &= \quad \min \sum_{j=1}^{m} t_{kj} \\ \text{s.t.} & \\ & \quad v_k - y_j \leq t_{kj},\ t_{kj} \geq 0 \quad \text{for } j = 1, 2, \ldots, m. \end{aligned} \tag{4.39}$$

Therefore, the entire lexicographic model (4.37) can be formulated as follows:

$$\begin{aligned} &\operatorname{lex\,min} \quad \left[\sum_{j=1}^{m} t_{2j},\ \sum_{j=1}^{m} t_{3j}, \ldots,\ \sum_{j=1}^{m} t_{rj} \right] \\ &\text{s.t.} \\ &\qquad v_k - f_j(\mathbf{x}) \leq t_{kj},\ t_{kj} \geq 0 \quad \text{for } j = 1, 2, \ldots, m, \quad k = 2, \ldots, r \\ &\qquad \mathbf{x} \in Q. \end{aligned} \tag{4.40}$$

Note that the above formulation, unlike the problem (4.35), does not use integer variables and can be considered as an LP modification of the original multiple criteria problem (2.1). Thus, this model preserves the problem's convexity when the original problem is defined with a convex feasible set Q and a concave objective functions f_j. The size of problem (4.40) depends on the number of different outcome values. Thus, for many problems with not too large number of outcome values, the problem can easily be solved directly and even for convex problems such an approach may be more efficient than the sequential algorithms discussed in the previous subsection. Note that in many problems of telecommunications network design, the objective functions express the quality of service and one can easily consider a limited finite scale (grid) of the corresponding outcome values. Similarly, in the capacity protection design (Section 3.3), one may focus on a finite grid of demand volumes. One may also notice that model (4.40) opens a way for the fuzzy representation of quality measures within the MMF problems.

5 Concluding remarks

Today, the major objective of telecommunications network design for Internet services is to maximize service data flows and provide fair treatment of all services. Fair treatment of services can be formalized through the MMF solution concept, which assumes that the worst service performance is maximized and the solution is additionally regularized with the lexicographic maximization of the second worst performance, the third one etc. We have argued that the MMF solution concept is tightly related to the Rawlsian principle of justice and is equivalent the lexicographic Max-Min concept.

We have shown that with respect to telecommunications networks carrying the so-called elastic traffic, the problems of routing design, restoration design and protection capacity design are examples of important design problems that can be formulated with the use of the MMF notion to express design objectives. We have presented and evaluated several general efficient sequential algorithms that can be used to solve the basic variants of these problems as well as many other MMF problems. These algorithms are based on the idea to solve a sequence of properly defined Max-Min subproblems. The algorithms differ with respect to the strategy of choosing this sequence. We have shown that the efficiency of different strategies depends on the distribution of outcome values of the optimal solution to the original problem. Since the algorithms can still be time-consuming due to excessive number of subproblems that have to be solved in the iteration process, the values of subproblems' dual variables can be used to considerably reduce the number of solved subproblems. In the case of LP problem formulations the values of dual variables can be obtained directly from the simplex tableau.

Unfortunately, sequential algorithms are only applicable to convex problems. Hence if network design problems are augmented with the requirements that data flows are to be routed along single paths or that link capacity is modular, these algorithms cannot be applied any more. However, we have shown that the original problem of lexicographic maximization of the solution vector can be replaced with the lexicographic minimization of the vector that describes the distribution of outcome values, which, fortunately enough, is convex as long as an original problem is defined with a convex feasible set Q and a concave objective functions f_j. The complexity of the transformed problem is directly related to the number of different outcome values. As far as telecommunications network design is concerned, this number can be pretty small, for example if the objective functions express the quality of service. Therefore, further research on application of distribution approach to various classes of telecommunications MMF problems seems to be very promising.

Appendix A: Numerical Example

In this appendix we present a numerical example of Problem 1 (Section 3.1). The structure of the considered network is shown in Figure 1; c_e denotes the capacity of link e. We assume that the set of demands corresponds to the set of all pairs of nodes.

The results of applying Algorithm 4 (Section 4.1) to Problem 1 are presented in Table 1. The table contains information pertaining to consecutive iterations of the algorithm. The information includes the number of demands blocked in an iteration and their flow size. To effectively solve the problem we applied a path (column) generation technique [27, Section 8.2.1] allowing for problem decomposition. The overall number of paths used in

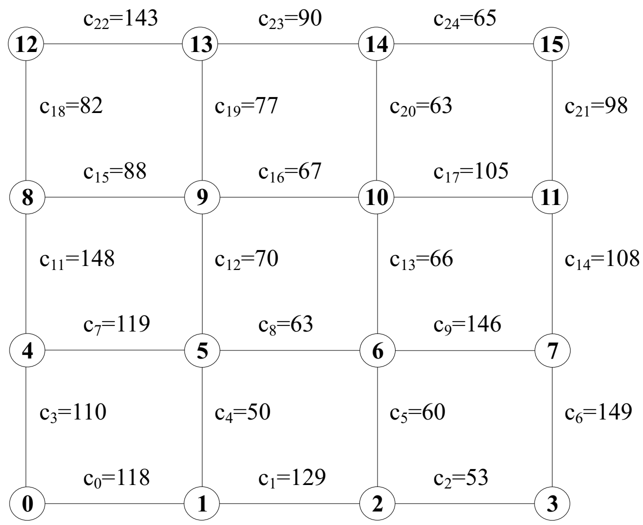

Figure 1: 16-node square network

Table 1: Consecutive values of t^* and number of blocked demands in MMF allocation procedure

Iteration n	**1**	**2**	**3**	**4**	**5**	**6**
t_n^*	5.286	6.625	7.013	10.214	14.606	16.115
Blocked Demands No.	63	8	28	8	4	1
Iteration n	**7**	**8**	**9**	**10**	**11**	**12**
t_n^*	25.362	29.908	30.962	35.093	49.288	82.145
Blocked Demands No.	1	2	1	2	1	1

each iteration is presented in Figure 2. The LP subproblems were solved with the use of the CPLEX 9.0 optimization package. Solving the problem on a PC-class computer equipped with a 2.4 GHz P4 HT processor required 0.2 s of the processor time, of which only 0.03 s in total was spent on solving the LP subproblems.

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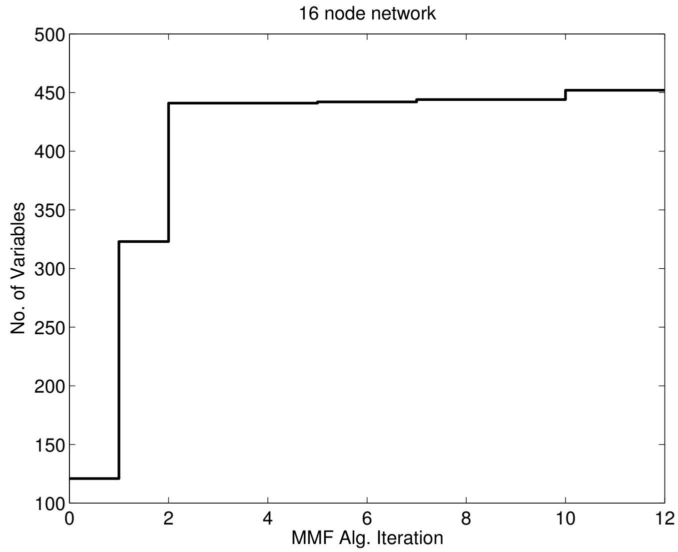

Figure 2: Number of problem columns in function of MMF algorithm iterations

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