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# Conditional Value at Risk and Related Linear Programming Models for Portfolio Optimization\*

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## Abstract

Many risk measures have been recently introduced which (for discrete random variables) result in Linear Programming (LP) models. While some LP computable risk measures may be viewed as approximations to the variance (e.g., the mean absolute deviation or the Gini's mean absolute difference), shortfall or quantile risk measures are recently gaining more popularity in various financial applications. In this paper we study LP solvable portfolio optimization models based on extensions of the Conditional Value at Risk (CVaR) measure. The models use multiple CVaR measures thus allowing for more detailed risk aversion modeling. We study both the theoretical properties of the models and their performance on real-life data.

**Key words.** Portfolio optimization, mean-risk and mean-safety model, linear programming, conditional value at risk, Gini's mean difference, multiple criteria, experimental analysis.

## 1 Introduction

Following the seminal work by Markowitz (1952), the portfolio optimization problem is modeled as a mean-risk bicriteria optimization problem where the expected return is maximized and some (scalar) risk measure is minimized. In the original Markowitz model the risk is measured by the standard deviation or variance. Several other risk measures have been later considered thus creating the entire family of mean-risk (Markowitz-type) models. While the original Markowitz model forms a quadratic programming problem, following Sharpe (1971a), many attempts have been made to linearize the portfolio optimization procedure (c.f., Speranza (1993) and references therein). The LP solvability is very important for applications to real-life financial decisions where the constructed portfolios have to meet numerous side constraints (including the minimum transaction lots (Mansini and Speranza (1999), Bonaglia et al. (2002)), transaction costs (Kellerer, Mansini and Speranza (2000), Konno and Wiyayanayake (2001)) and mutual funds characteristics (Chiodi, Mansini and Speranza (2003))). The introduction of these features

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leads to mixed integer LP problems. In order to guarantee that the portfolio takes advantage of diversification, no risk measure can be a linear function of the portfolio weights. Nevertheless, a risk measure can be LP computable in the case of discrete random variables, i.e., in the case of returns defined by their realizations under specified scenarios.

The simplest LP computable risk measures are dispersion measures similar to the variance. The mean absolute deviation was very early considered in portfolio analysis (Sharpe (1971b) and references therein) while Konno and Yamazaki (1991) presented and analyzed the complete portfolio optimization model (the so-called MAD model). Yitzhaki (1982) introduced the mean-risk model using Gini's mean (absolute) difference as the risk measure. Both the mean absolute deviation and the Gini's mean difference turn out to be special aggregation techniques of the multiple criteria LP model (Ogryczak (2000)) based on the pointwise comparison of the absolute Lorenz curves. The latter leads the quantile shortfall risk measures which are more commonly used and accepted. Recently, the second order quantile risk measures have been introduced in different ways by many authors (Artzner et al. (1999), Ogryczak (1999), Rockafellar and Uryasev (2000)). The measure, now commonly called the Conditional Value at Risk (CVaR) (after Rockafellar and Uryasev (2000)) or Tail VaR, represents the mean shortfall at a specified confidence level. It leads to LP solvable portfolio optimization models in the case of discrete random variables represented by their realizations under specified scenarios. The CVaR has been shown by Pflug (2000) to satisfy the requirements of the so-called coherent risk measures (Artzner et al. (1999)) and is consistent with the second degree stochastic dominance as shown by Ogryczak and Ruszczyński (2002a). Several empirical analyses (Andersson et al. (2001), Rockafellar and Uryasev (2002), Mansini, Ogryczak and Speranza (2003b), Topaloglou, Vladimirov and Zenios (2002)) confirm its applicability to various financial optimization problems. Thus, the CVaR models seem to overstep the measure of Value-at-Risk (VaR) defined as the maximum loss at a specified confidence level which is commonly used in banking (c.f., Jorion (2001) and references therein).

This paper deals with portfolio optimization models based on the use of multiple CVaR risk measures. Such an extension allows for more detailed risk aversion modeling while preserving the simplicity of the original CVaR model. Both the theoretical properties of the models and their performance on real data are analyzed. The paper is organized as follows. In the next section we introduce basics of the mean-risk portfolio optimization, the CVaR risk measures and the concepts necessary to make the paper self-contained. Section 3 is devoted to the extended multiple CVaR model. Our analysis has been focused on the weighted CVaR (WCVaR) measures defined as simple combinations of a few CVaR measures. The general model is presented and its two specific weights setting schemes relating the WCVaR measure to the Gini's mean difference and its tail version, respectively, are analyzed in detail. Moreover, a CVaR-related LP technique to directly enforce portfolio diversification is introduced. Section 4 presents the experimental analysis on real data from the Milan Stock Exchange. Extensive in-sample and out-of-sample computational results are provided and commented. Finally, some concluding remarks are given.

## 2 Basic models

### 2.1 Mean-safety portfolio optimization

At the beginning of a period, an investor allocates the capital among various securities, thus assigning a nonnegative weight (share of the capital) to each security. Let  $J = \{1, 2, \dots, n\}$  denote a set of securities considered for investment. For each security  $j \in J$ , its rate of return is represented by a random variable  $R_j$  with a given mean  $\mu_j = \mathbb{E}\{R_j\}$ . Further, let  $\mathbf{x} = (x_j)_{j=1,2,\dots,n}$  denote a vector of decision variables  $x_j$  expressing the weights defining a portfolio. To represent a portfolio, the weights must satisfy a set

of constraints that form a feasible set  $\mathcal{P}$ . The simplest way of defining a feasible set is by a requirement that the weights must sum to one and short sales are not allowed, i.e.  $\sum_{j=1}^n x_j = 1$  and  $x_j \geq 0$  for  $j = 1, \dots, n$ . Hereafter, it is assumed that  $\mathcal{P}$  is a general LP feasible set given in a canonical form as a system of linear equations with nonnegative variables: This allows one to include upper bounds on single shares as well as several more complex portfolio structure restrictions which may be faced by a real-life investor.

Each portfolio  $\mathbf{x}$  defines a corresponding random variable  $R_{\mathbf{x}} = \sum_{j=1}^n R_j x_j$  that represents the portfolio rate of return. We consider  $T$  scenarios with probabilities  $p_t$  (where  $t = 1, \dots, T$ ). We assume that for each random variable  $R_j$  its realization  $r_{jt}$  under the scenario  $t$  is known. Typically, the realizations are derived from historical data treating  $T$  historical periods as equally probable scenarios ( $p_t = 1/T$ ). The realizations of the portfolio return  $R_{\mathbf{x}}$  are given as  $y_t = \sum_{j=1}^n r_{jt} x_j$  and the expected value can be computed as  $\mu(\mathbf{x}) = \sum_{t=1}^T y_t p_t = \sum_{t=1}^T \left[ \sum_{j=1}^n r_{jt} x_j \right] p_t$ . Similarly, several risk measures can be LP computable with respect to the realizations  $y_t$ .

Following Markowitz (1952), the portfolio optimization problem is modeled as a mean-risk bicriteria optimization problem where the mean  $\mu(\mathbf{x})$  is maximized and the risk measure  $\varrho(\mathbf{x})$  is minimized. In the original Markowitz model, the standard deviation  $\sigma(\mathbf{x}) = [\mathbb{E}\{(R_{\mathbf{x}} - \mu(\mathbf{x}))^2\}]^{1/2}$  was used as the risk measure. Several other risk measures have been later considered thus creating the entire family of mean-risk models (see Mansini, Ogryczak and Speranza (2003a, 2003b)). These risk measures, similar to the standard deviation, are not affected by any shift of the outcome scale and are equal to 0 in the case of a risk-free portfolio while taking positive values for any risky portfolio. Unfortunately, such risk measures are not consistent with the stochastic dominance order (Whitmore and Findlay (1978)) or other axiomatic models of risk-averse preferences (Rothschild and Stiglitz (1969) and risk measurement (Artzner et al. (1999))).

In stochastic dominance, uncertain returns (modeled as random variables) are compared by point-wise comparison of some performance functions constructed from their distribution functions. The first performance function  $F_{\mathbf{x}}^{(1)}$  is defined as the right-continuous cumulative distribution function:  $F_{\mathbf{x}}^{(1)}(\eta) = F_{\mathbf{x}}(\eta) = \mathbb{P}\{R_{\mathbf{x}} \leq \eta\}$  and it defines the first degree stochastic dominance (FSD). The second function is derived from the first as  $F_{\mathbf{x}}^{(2)}(\eta) = \int_{-\infty}^{\eta} F_{\mathbf{x}}(\xi) d\xi$  and it defines the (weak) relation of *second degree stochastic dominance* (SSD):  $R_{\mathbf{x}'} \succeq_{SSD} R_{\mathbf{x}''}$  if  $F_{\mathbf{x}'}^{(2)}(\eta) \leq F_{\mathbf{x}''}^{(2)}(\eta)$  for all  $\eta$ . We say that portfolio  $\mathbf{x}'$  *dominates*  $\mathbf{x}''$  *under the SSD* ( $R_{\mathbf{x}'} \succ_{SSD} R_{\mathbf{x}''}$ ), if  $F_{\mathbf{x}'}^{(2)}(\eta) \leq F_{\mathbf{x}''}^{(2)}(\eta)$  for all  $\eta$ , with at least one strict inequality. A feasible portfolio  $\mathbf{x}^0 \in \mathcal{P}$  is called *SSD efficient* if there is no  $\mathbf{x} \in \mathcal{P}$  such that  $R_{\mathbf{x}} \succ_{SSD} R_{\mathbf{x}^0}$ .

Several other portfolio performance measures were introduced as safety measures to be maximized, like the worst realization, analyzed by Young (1998), and the CVaR risk measures we consider further. Contrary to risk measures, the safety measures may be consistent with formal models of risk-averse preferences (Rothschild and Stiglitz (1969)) and risk measurement (Artzner et al. (1999)). It has been shown by Mansini, Ogryczak and Speranza (2003a, 2003b) that for any risk measure  $\varrho(\mathbf{x})$  a corresponding *safety* measure  $\mu_{\varrho}(\mathbf{x}) = \mu(\mathbf{x}) - \varrho(\mathbf{x})$  can be defined and viceversa. Note that while risk measure  $\varrho(\mathbf{x})$  is a convex function of  $\mathbf{x}$ , the corresponding safety measure  $\mu_{\varrho}(\mathbf{x})$  is concave. A safety measure is considered risk relevant if for any risky portfolio its value is less than the value for the risk-free portfolio with the same expected returns. We say that the safety measure  $\mu_{\varrho}(\mathbf{x})$  is *SSD consistent* (or that the risk measure  $\varrho(\mathbf{x})$  is *SSD safety consistent*) if  $R_{\mathbf{x}'} \succeq_{SSD} R_{\mathbf{x}''}$  implies  $\mu_{\varrho}(\mathbf{x}') \geq \mu_{\varrho}(\mathbf{x}'')$ . If the safety measure is SSD consistent, then except for portfolios with identical values of  $\mu(\mathbf{x})$  and  $\mu_{\varrho}(\mathbf{x})$  (and thereby  $\varrho(\mathbf{x})$ ),

every efficient solution of the bicriteria problem

$$\max\{[\mu(\mathbf{x}), \mu_\rho(\mathbf{x})] : \mathbf{x} \in \mathcal{P}\} \quad (1)$$

is an SSD efficient portfolio (Ogryczak and Ruszczyński (1999)). Therefore, we will focus on the mean-safety bicriteria optimization (1) rather than on the classical mean-risk model.

The commonly accepted approach to implement the Markowitz-type mean-risk models is based on the use of a specified lower bound  $\mu_0$  on expected return while minimizing the risk criterion. In our analysis we use the bounding approach applied to the maximization of the safety measures, i.e.

$$\max\{\mu_\rho(\mathbf{x}) : \mathbf{x} \in \mathcal{P}, \mu(\mathbf{x}) \geq \mu_0\}. \quad (2)$$

For small values of the bound  $\mu_0$ , the constraint  $\mu(\mathbf{x}) \geq \mu_0$  does not influence the optimization (2). In this case, the portfolio obtained is the so called Maximum Safety Portfolio (MSP), whose return is referred to as  $\mu(\text{MSP})$ . The MSP is the solution of  $\max_{\mathbf{x} \in \mathcal{P}} \mu_\rho(\mathbf{x})$ . When  $\mu_0 \geq \mu(\text{MSP})$ , then the optimal solution of the corresponding problem represents a mean-safety efficient solution. In our computational analysis we will examine the MSPs for the different models. We will obtain the MSPs by solving (2), with  $\mu_0$  set to zero.

## 2.2 Absolute Lorenz curve and related measures

Stochastic dominance relates the notion of risk to a possible failure of achieving some targets. Note that function  $F_{\mathbf{x}}^{(2)}$ , used to define the SSD relation, can also be presented as follows (Ogryczak and Ruszczyński (1999, 2001)):  $F_{\mathbf{x}}^{(2)}(\eta) = \mathbb{E}\{\max\{\eta - R_{\mathbf{x}}, 0\}\}$  and its values are LP computable for returns represented by their realizations  $y_t$  as:

$$F_{\mathbf{x}}^{(2)}(\eta) = \min \sum_{t=1}^T d_t^- p_t \quad \text{subject to} \quad d_t^- \geq \eta - y_t, \quad d_t^- \geq 0 \quad \text{for } t = 1, \dots, T. \quad (3)$$

In this paper we focus on quantile shortfall risk measures related to the so-called *Absolute Lorenz Curves* (ALC) (Levy and Kroll (1978), Shorrocks (1983), Shalit and Yitzhaki (1994), Ogryczak (1999), Ogryczak and Ruszczyński (2002a)) which represent the second quantile functions defined as

$$F_{\mathbf{x}}^{(-2)}(p) = \int_0^p F_{\mathbf{x}}^{(-1)}(\alpha) d\alpha \quad \text{for } 0 < p \leq 1 \quad \text{and} \quad F_{\mathbf{x}}^{(-2)}(0) = 0, \quad (4)$$

where  $F_{\mathbf{x}}^{(-1)}(p) = \inf \{\eta : F_{\mathbf{x}}(\eta) \geq p\}$  is the left-continuous inverse of the cumulative distribution function  $F_{\mathbf{x}}$ . Actually, the pointwise comparison of ALCs provides an alternative characterization of the SSD relation (Ogryczak and Ruszczyński (2002a)) in the sense that  $R_{\mathbf{x}'} \succeq_{SSD} R_{\mathbf{x}''}$  if and only if  $F_{\mathbf{x}'}^{(-2)}(\beta) \geq F_{\mathbf{x}''}^{(-2)}(\beta)$  for all  $0 < \beta \leq 1$ . The duality (conjugency) relation between  $F^{(-2)}$  and  $F^{(2)}$  (Ogryczak (1999), Ogryczak and Ruszczyński (2002a)) leads to the following formula:

$$F_{\mathbf{x}}^{(-2)}(\beta) = \max_{\eta \in \mathbb{R}} \left[ \beta \eta - F_{\mathbf{x}}^{(2)}(\eta) \right] = \max_{\eta \in \mathbb{R}} \left[ \beta \eta - \mathbb{E}\{\max\{\eta - R_{\mathbf{x}}, 0\}\} \right] \quad (5)$$

where  $\eta$  is a real variable taking the value of  $\beta$ -quantile  $Q_{\beta}(\mathbf{x})$  at the optimum.

For any real tolerance level  $0 < \beta \leq 1$ , the normalized value of the ALC defined as  $M_{\beta}(\mathbf{x}) = F_{\mathbf{x}}^{(-2)}(\beta)/\beta$  is now commonly called the *Conditional Value-at-Risk (CVaR)*. This name was introduced

by Rockafellar and Uryasev (2000) who considered (similar to the Expected Shortfall by Embrechts, Klüppelberg and Mikosch (1997)) the measure CVaR defined as  $\mathbb{E}\{R_{\mathbf{x}} | R_{\mathbf{x}} \leq F_{\mathbf{x}}^{(-1)}(\beta)\}$  for continuous distributions showing that it could then be expressed by a formula analogous to (5) and thus potentially LP computable. The approach has been further expanded to general distributions (Rockafellar and Uryasev (2002)). For additional discussion of relations between various definitions of the measures we refer to (Ogryczak and Ruszczyński (2002b)).

The CVaR measure is a safety measure according to our classification (Mansini, Ogryczak and Speranza (2003a)). The corresponding risk measure  $\Delta_{\beta}(\mathbf{x}) = \mu(\mathbf{x}) - M_{\beta}(\mathbf{x})$  (Ogryczak and Ruszczyński (2002b)) is called hereafter the (worst) *conditional semideviation*. Note that, for any  $0 < \beta < 1$ , the CVaR measures defined by  $F^{(-2)}(\beta)$ , opposite to below-target mean deviations  $F^{(2)}(\eta)$ , are risk relevant. They are also coherent (Pflug (2000) and SSD consistent (Ogryczak and Ruszczyński (2002a)). For a discrete random variable represented by its realizations  $y_t$ , due to (3), problem (5) becomes an LP. Thus

$$M_{\beta}(\mathbf{x}) = \max \left[ \eta - \frac{1}{\beta} \sum_{t=1}^T d_t^- p_t \right] \quad \text{s.t.} \quad d_t^- \geq \eta - y_t, \quad d_t^- \geq 0 \quad \text{for } t = 1, \dots, T. \quad (6)$$

The CVaR measure is an increasing function of the tolerance level  $\beta$ , with  $M_1(\mathbf{x}) = \mu(\mathbf{x})$ . For  $\beta$  approaching 0, the CVaR measure tends to the Minimax safety measure (Young (1998))

$$M(\mathbf{x}) = \min_{t=1, \dots, T} y_t \quad (7)$$

whose corresponding risk measure is

$$\Delta(\mathbf{x}) = \mu(\mathbf{x}) - M(\mathbf{x}). \quad (8)$$

One may notice that  $\Delta_{0.5}(\mathbf{x})$  represents the mean absolute deviation from the median (Mansini, Ogryczak and Speranza (2003a)), the risk measure suggested by Sharpe (1971b) as the right MAD model.

Yitzhaki (1982) introduced the GMD model using Gini's mean (absolute) difference as the risk measure. For a discrete random variable represented by its realizations  $y_t$ , the *Gini's mean difference*  $\Gamma(\mathbf{x}) = \frac{1}{2} \sum_{t'=1}^T \sum_{t''=1}^T |y_{t'} - y_{t''}| p_{t'} p_{t''}$  is LP computable (when minimized). Actually, Yitzhaki (1982) suggested to use the corresponding safety measure

$$\mu_{\Gamma}(\mathbf{x}) = \mu(\mathbf{x}) - \Gamma(\mathbf{x}) = \mathbb{E}\{R_{\mathbf{x}} \wedge R_{\mathbf{x}}\} \quad (9)$$

to take advantages of its SSD consistency. The measure is LP computable as:

$$\begin{aligned} \mu_{\Gamma}(\mathbf{x}) &= \max \sum_{t=1}^T p_t^2 y_t + 2 \sum_{t'=1}^{T-1} \sum_{t''=t'+1}^T p_{t'} p_{t''} u_{t' t''} \\ \text{s.t.} \quad &u_{t' t''} \leq y_{t'}, \quad u_{t' t''} \leq y_{t''} \quad \text{for } t' = 1, \dots, T-1; \quad t'' = t'+1, \dots, T. \end{aligned} \quad (10)$$

Both the Gini's mean difference and the CVaR measures are related to the absolute Lorenz curve (4). The graph of  $F_{\mathbf{x}}^{(-2)}$  is a continuous convex curve connecting points  $(0, 0)$  and  $(1, \mu(\mathbf{x}))$ , whereas a deterministic outcome with the same expected value  $\mu(\mathbf{x})$ , yields the chord (straight line) connecting the same points. Hence, the space between the curve  $(p, F_{\mathbf{x}}^{(-2)}(p))$ ,  $0 \leq p \leq 1$ , and its chord represents the dispersion (and thereby the riskiness) of  $R_{\mathbf{x}}$  in comparison to the deterministic outcome of  $\mu(\mathbf{x})$ . It

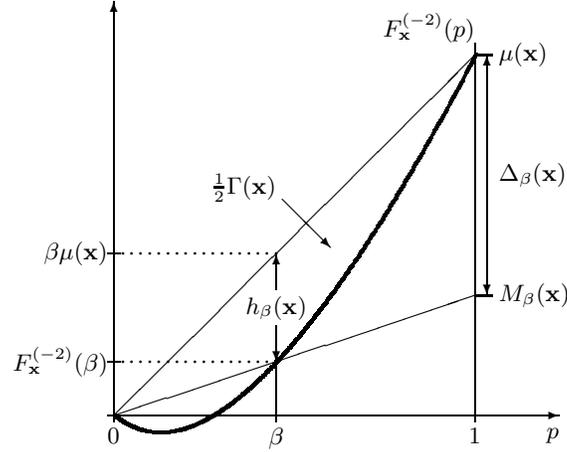


Figure 1: The absolute Lorenz curve and risk measures.

is called the *Lorenz dispersion space*. One may notice that  $\Delta_\beta(\mathbf{x}) = \frac{1}{\beta}h_\beta(\mathbf{x})$  where  $h_\beta(\mathbf{x})$  denotes the vertical diameter of the Lorenz dispersion space at point  $p = \beta$  (Fig. 1).

Actually, all the classical LP computable risk measures are well defined size characteristics of the Lorenz dispersion space (Ogryczak (2000), Ogryczak and Ruszczyński (2002a)). The Gini's mean difference may be expressed as  $\Gamma(\mathbf{x}) = 2 \int_0^1 (\mu(\mathbf{x})\alpha - F_{\mathbf{x}}^{(-2)}(\alpha))d\alpha = 2 \int_0^1 h_\alpha(\mathbf{x})d\alpha$  thus representing the doubled area of the Lorenz dispersion space. Therefore, any CVaR measure (the conditional semideviation) is a rather rough (diameter) measure of the Lorenz dispersion space when comparing to the Gini's mean difference. The GMD safety measure summarizes all the CVaR measures as  $\mu_\Gamma(\mathbf{x}) = \mu(\mathbf{x}) - \Gamma(\mathbf{x}) = 2 \int_0^1 F_{\mathbf{x}}^{(-2)}(\alpha)d\alpha = 2 \int_0^1 \alpha M_\alpha(\mathbf{x})d\alpha$ . Therefore, the stronger SSD consistency results have been recently shown for the GMD model by Ogryczak and Ruszczyński (2002a), i.e.,  $R_{\mathbf{x}'} \succ_{SSD} R_{\mathbf{x}''}$  implies  $\mu_\Gamma(\mathbf{x}') > \mu_\Gamma(\mathbf{x}'')$  which guarantees that every efficient solution of the bicriteria problem (1) is an SSD efficient portfolio. On the other hand, its computational LP model (10) requires  $T^2$  variables which makes it much more complicated than the CVaR model (6) using only  $T$  variables. In the next sections we will demonstrate that models based on a few CVaR criteria offer a very good compromise between the computationally complex GMD model and simplified CVaR.

### 3 Enhanced CVaR measures

#### 3.1 The multiple CVaR model

Although any CVaR measure (for  $0 < \beta < 1$ ) is risk relevant, it represents only the mean within a part (tail) of the distribution of returns. Therefore, such a single criterion is in some manner crude for modeling various risk aversion preferences. In order to enrich the modeling capabilities, one needs to treat differently some more or less extreme events (Haimes (1993)). For this purpose one may consider multiple CVaR measures as risk (safety) criteria. In particular, one may consider several, say  $m$ , tolerance levels  $0 < \beta_1 < \beta_2 < \dots < \beta_m \leq 1$  and use the corresponding CVaR measures  $M_{\beta_k}(\mathbf{x})$  to build a

multiple criteria portfolio selection model:

$$\max\{[M_{\beta_1}(\mathbf{x}), M_{\beta_2}(\mathbf{x}), \dots, M_{\beta_m}(\mathbf{x})] : \mathbf{x} \in \mathcal{P}\}. \quad (11)$$

The model may contain the original mean value as the last criterion  $M_1(\mathbf{x}) = \mu(\mathbf{x})$ , if  $\beta_m = 1$ . One may notice that, for any portfolio  $\mathbf{x}$ , one gets  $[M_{\beta_1}(\mathbf{x}), M_{\beta_2}(\mathbf{x}), \dots, M_{\beta_m}(\mathbf{x})] \leq [\mu(\mathbf{x}), \mu(\mathbf{x}), \dots, \mu(\mathbf{x})]$  with at least one inequality strict. Hence, the multiple criteria model (11) is risk relevant in the sense that for any risky portfolio its outcome vector is dominated by that for the risk-free portfolio with the same expected return. Actually, the model (11) is SSD consistent in the sense that  $R_{\mathbf{x}'} \succeq_{SSD} R_{\mathbf{x}''}$  implies  $[M_{\beta_1}(\mathbf{x}'), M_{\beta_2}(\mathbf{x}'), \dots, M_{\beta_m}(\mathbf{x}')] \geq [M_{\beta_1}(\mathbf{x}''), M_{\beta_2}(\mathbf{x}''), \dots, M_{\beta_m}(\mathbf{x}'')]$ . Actually, the following assertion is valid.

**Theorem 1** *For any set of levels  $0 < \beta_1 < \beta_2 < \dots < \beta_m \leq 1$ , except for portfolios with identical values of all the corresponding CVaR measures  $M_{\beta_k}(\mathbf{x})$ , every efficient solution of the multiple criteria problem (11) is an SSD efficient portfolio.*

**Proof.** Let  $\mathbf{x}^0 \in \mathcal{P}$  be an efficient solution of (11). Suppose that there exists  $\mathbf{x}' \in \mathcal{P}$  such that  $R_{\mathbf{x}'} \succ_{SSD} R_{\mathbf{x}^0}$ . Then, due to SSD consistency of the CVaR measures,  $M_{\beta_k}(\mathbf{x}') \geq M_{\beta_k}(\mathbf{x}^0)$  for all  $k = 1, \dots, m$ . The latter together with the fact that  $\mathbf{x}^0$  is efficient, implies that  $M_{\beta_k}(\mathbf{x}') = M_{\beta_k}(\mathbf{x}^0)$  for  $k = 1, \dots, m$ , which completes the proof.  $\square$

The weighted sum is the simplest aggregation technique in multiple criteria optimization. It can also be used to combine the CVaR criteria in (11). The weighted CVaR objective was first introduced by Ogryczak (2000) (not using the name CVaR introduced later by Rockafellar and Uryasev (2000)); the portfolio optimization model based on historical data and its LP computability was then proven. Later it was extended and considered in various forms for portfolio optimization (Ogryczak and Ruszczyński, 2002b; Acerbi and Simonetti, 2002), general decisions under risk (Ogryczak, 2002), as well as for regression analysis (Rockafellar, Uryasev and Zabarankin, 2002).

In order to distinguish clearly the  $\mu(\mathbf{x})$  criterion, further we will consider it separately from the  $m$  tolerance levels  $0 < \beta_1 < \beta_2 < \dots < \beta_m < 1$  (thus excluding  $\beta = 1$ ). Combining  $\mu(\mathbf{x})$  and the CVaR values  $M_{\beta_k}(\mathbf{x})$  with positive (and normalized) weights we introduce the Weighted multiple CVaR (WCVaR) measure as

$$\begin{aligned} M_{\mathbf{w}}^{(m)}(\mathbf{x}) &= w_0 \mu(\mathbf{x}) + \sum_{k=1}^m w_k M_{\beta_k}(\mathbf{x}) \\ \sum_{k=0}^m w_k &= 1, \quad w_0 \geq 0, \quad w_k > 0 \quad \text{for } k = 1, \dots, m. \end{aligned} \quad (12)$$

The WCVaR measure is, obviously, a safety measure and it is risk relevant. The corresponding risk measure turns out to be the weighted sum of the  $\Delta_{\beta_k}(\mathbf{x})$  measures thus forming the weighted conditional semideviation:

$$\Delta_{\mathbf{w}}^{(m)}(\mathbf{x}) = \mu(\mathbf{x}) - M_{\mathbf{w}}^{(m)}(\mathbf{x}) = \sum_{k=1}^m w_k \Delta_{\beta_k}(\mathbf{x}), \quad \sum_{k=1}^m w_k \leq 1, \quad w_k > 0 \quad \text{for } k = 1, \dots, m. \quad (13)$$

The latter is not affected by any shift of the outcome scale and it is equal to 0 in the case of a risk-free portfolio while taking positive value for any risky portfolio, thus representing a translation invariant and

risk relevant dispersion parameter. Therefore, we can consider the corresponding Markowitz-type model and its mean-safety formalization (1):

$$\max\{[\mu(\mathbf{x}), M_{\mathbf{w}}^{(m)}(\mathbf{x})] : \mathbf{x} \in \mathcal{P}\} = \max\{[\mu(\mathbf{x}), \mu(\mathbf{x}) - \Delta_{\mathbf{w}}^{(m)}(\mathbf{x})] : \mathbf{x} \in \mathcal{P}\}. \quad (14)$$

Since the CVaR measures are coherent (Pflug, 2000) and SSD consistent (Ogryczak and Ruszczyński (2002a)), the same applies to the WCVaR measure. In particular,  $R_{\mathbf{x}'} \succeq_{SSD} R_{\mathbf{x}''}$  implies  $M_{\mathbf{w}}^{(m)}(\mathbf{x}') \geq M_{\mathbf{w}}^{(m)}(\mathbf{x}'')$  (Ogryczak and Ruszczyński (2002b)). Actually, the SSD consistency relation for the WCVaR measure is stronger since it takes into account all the individual CVaR measures as shown in the following assertion.

**Theorem 2** *For any set of levels  $0 < \beta_1 < \beta_2 < \dots < \beta_m \leq 1$ , except for portfolios with identical values of  $\mu(\mathbf{x})$  and all conditional semideviations  $\Delta_{\beta_k}(\mathbf{x})$ , respectively, every efficient solution of the bicriteria problem (14) is an SSD efficient portfolio.*

**Proof.** Let  $\mathbf{x}^0 \in \mathcal{P}$  be an efficient solution of (14). Suppose that there exists  $\mathbf{x}' \in \mathcal{P}$  such that  $R_{\mathbf{x}'} \succ_{SSD} R_{\mathbf{x}^0}$ . Then, due to SSD consistency of the CVaR measures,  $\mu(\mathbf{x}') \geq \mu(\mathbf{x}^0)$  and  $M_{\beta_k}(\mathbf{x}') \geq M_{\beta_k}(\mathbf{x}^0)$  for all  $k = 1, \dots, m$ . The latter, together with the fact that  $\mathbf{x}^0$  is efficient, implies that  $\mu(\mathbf{x}') = \mu(\mathbf{x}^0)$  and  $\sum_{k=1}^m w_k M_{\beta_k}(\mathbf{x}') = \sum_{k=1}^m w_k M_{\beta_k}(\mathbf{x}^0)$ . Hence,  $M_{\beta_k}(\mathbf{x}') = M_{\beta_k}(\mathbf{x}^0)$  for  $k = 1, \dots, m$ , and therefore,  $\Delta_{\beta_k}(\mathbf{x}') = \Delta_{\beta_k}(\mathbf{x}^0)$  for all  $k = 1, \dots, m$ , which completes the proof.  $\square$

For returns represented by their realizations we get an LP optimization problem. The model contains the following core LP constraints to define a feasible portfolio, portfolio realizations, and portfolio expected return:

$$\begin{aligned} \mathbf{x} \in \mathcal{P} & \quad \text{and} \quad z \geq \mu_0, \\ \sum_{j=1}^n \mu_j x_j = z & \quad \text{and} \quad \sum_{j=1}^n r_{jt} x_j = y_t \quad \text{for } t = 1, \dots, T \end{aligned} \quad (15)$$

where  $z$  is an unbounded variable representing the mean return of the portfolio  $\mathbf{x}$  and  $y_t$  ( $t = 1, \dots, T$ ) are unbounded variables to represent the realizations of the portfolio return under the scenario  $t$ .

The general WCVaR model (14) leads us to the following LP problem:

$$\begin{aligned} \text{maximize} \quad & w_0 z + \sum_{k=1}^m w_k q_k - \sum_{k=1}^m \frac{w_k}{\beta_k} \sum_{t=1}^T p_t d_{tk} \\ \text{subject to} \quad & (15) \text{ and } d_{tk} - q_k + y_t \geq 0, \quad d_{tk} \geq 0 \quad \text{for } t = 1, \dots, T; \quad k = 1, \dots, m \end{aligned} \quad (16)$$

where  $q_k$  (for  $k = 1, \dots, m$ ) are unbounded variables taking the values of the corresponding  $\beta_k$ -quantiles (in the optimal solution). Except from the core constraints (15), model (16) contains  $T$  nonnegative variables  $d_{tk}$  and  $T$  corresponding linear inequalities for each  $k$ . Thus, its dimensionality is proportional to the number of scenarios  $T$  and to the number of tolerance levels  $m$ . Note that model (16) with  $m = 1$  and  $w_0 = 0$  covers the standard CVaR model, while  $m > 1$  and various settings of positive weights  $w_k$  allow us to model a wide gamut of risk averse preferences. The model itself does not require any specific relation between the number of scenarios  $T$  and the number of securities  $n$  or the number of tolerance levels  $m$ . Certainly, similar to the Markowitz model, a very low number of scenarios may result in much less diversified portfolios. Increasing the number of tolerance levels  $m$ , generally, enables a larger diversification. However, such diversification is not guaranteed since, as demonstrated later, it also depends on a specific weights setting.

Recall that the absolute Lorenz curve, and thereby the CVaR measures, represent a dual characterization of the SSD relation (Ogryczak and Ruszczyński (2002a)). Hence, the weighted combination of the CVaR measures may be interpreted as the dual utility criterion within the theory developed by Yaari (1987) which was recently reintroduced into the finance literature in a simplified form of the spectral risk measures (Acerbi, 2002). Indeed, according to (12),

$$M_{\mathbf{w}}^{(m)}(\mathbf{x}) = w_0 \int_0^1 F_{\mathbf{x}}^{(-1)}(\alpha) d\alpha + \sum_{k=1}^m \frac{w_k}{\beta_k} \int_0^{\beta_k} F_{\mathbf{x}}^{(-1)}(\alpha) d\alpha = \int_0^1 \phi(\alpha) F_{\mathbf{x}}^{(-1)}(\alpha) d\alpha$$

where

$$\phi(\alpha) = \begin{cases} w_0 + \sum_{k=1}^m \frac{w_k}{\beta_k}, & 0 < \alpha \leq \beta_1 \\ w_0 + \sum_{k=i}^m \frac{w_k}{\beta_k}, & \beta_{i-1} < \alpha \leq \beta_i \\ w_0, & \beta_m < \alpha \leq 1 \end{cases} \quad (17)$$

is a decreasing risk aversion function (note the sign change for our safety measures to be maximized).

As pointed out by Acerbi (2002), the subjective risk aversion of an investor can be encoded in a function  $\phi(\alpha)$  defined for all possible confidence levels  $\alpha \in (0, 1]$  and from a financial point of view one cannot see any natural choice of function  $\phi(\alpha)$ . The use of a wide class of risk aversion functions in portfolio optimization (Acerbi and Simonetti (2002)) seems to be rather far from the simplicity necessary to make possible an effective implementation of the portfolio optimization procedure. In the following we will focus on the WCVaR measures defined as simple combinations of a very few CVaR measures (thus stepwise risk aversion functions  $\phi$  with a very few steps). On the other hand, we introduce two specific types of weights settings which relate the WCVaR measure to the Gini's mean difference and its tail version. This allows us to use a few tolerance levels  $\beta_k$  as the only parameters specifying the entire WCVaR measures (modeling risk aversion function) while the corresponding weights are automatically predefined by the requirements of the corresponding Gini's measures. In other words, the investor's preferences are modeled by a selection of a few tolerance levels. It turns out that we have managed to identify a class of simple WCVaR measures performing better in a real-life portfolio optimization environment than typical CVaR measures and the GMD model itself.

### 3.2 Wide WCVaR measures

In the case of equally probable  $T$  scenarios with  $p_t = 1/T$  (historical data for  $T$  periods), the weighted CVaR measure  $M_{\mathbf{w}}^{(T-1)}(\mathbf{x})$  defined with  $m = T - 1$  tolerance levels  $\beta_k = k/T$  for  $k = 1, 2, \dots, T - 1$  represents the standard weighting approach in the multiple criteria LP portfolio optimization model with criteria  $F^{(-2)}(k/T)$  (Ogryczak (2000)). The use of weights  $w_k = (2k)/T^2$  for  $k = 1, 2, \dots, T - 1$  and  $w_0 = 1/T$  results then in  $\Delta_{\mathbf{w}}^{(T-1)}(\mathbf{x}) = \frac{2}{T} \sum_{k=1}^{T-1} h_{\frac{k}{T}}(\mathbf{x}) = \Gamma(\mathbf{x})$  (c.f. Fig. 1). Hence, the WCVaR model is then equivalent to the GMD model. One may notice that LP formulation (16), similarly to the GMD one, requires  $T^2$  linear inequalities. Hence, this very specific model cannot provide us with any new modeling capabilities while causing a significant computational burden due to the large number of CVaRs.

In the general case of  $T$  scenarios with arbitrary probabilities  $p_t$ , one may use an approximation to  $\Gamma(\mathbf{x})$  with  $\Delta_{\mathbf{w}}^{(m)}(\mathbf{x})$  based on some reasonably chosen grid of tolerance levels  $\beta_k$ ,  $k = 1, \dots, m$

and weights  $w_k$  expressing the corresponding trapezoidal approximation to the integral formula  $\Gamma(\mathbf{x}) = 2 \int_0^1 (\mu(\mathbf{x})\alpha - F_{\mathbf{x}}^{(-2)}(\alpha))d\alpha$ . Such an approximation is a very attractive risk measure itself as it allows us to dramatically reduce the computational burden caused by  $T^2$  dimensionality of the LP implementation of the GMD model (10) while introducing new modeling capabilities connected to the grid selection. Exactly, for any grid of  $m$  tolerance levels  $0 < \beta_1 < \dots < \beta_k < \dots < \beta_m < 1$  one gets the trapezoidal approximation:

$$\Gamma(\mathbf{x}) \cong \sum_{k=1}^m (\beta_{k+1} - \beta_{k-1})h_{\beta_k}(\mathbf{x}) = \sum_{k=1}^m (\beta_{k+1} - \beta_{k-1})\beta_k \Delta_{\beta_k}(\mathbf{x})$$

where  $\beta_0 = 0$  and  $\beta_{m+1} = 1$ . Note that  $\sum_{k=1}^m (\beta_{k+1} - \beta_{k-1})\beta_k = \beta_m < 1$ . This leads us to the weighted CVaR measure with weights defined as:

$$w_k = (\beta_{k+1} - \beta_{k-1})\beta_k, \quad \text{for } k = 1, \dots, m, \quad \text{and} \quad w_0 = 1 - \beta_m. \quad (18)$$

Precisely, when using the weights given by (18), the corresponding WCVaR measure defined by (12) is an approximation to the GMD safety measure (9) (i.e.,  $M_{\mathbf{w}}^{(m)}(\mathbf{x}) \cong \mu_{\Gamma}(\mathbf{x})$ ), and the corresponding weighted conditional semideviation (13) is an approximation to the Gini's mean difference itself (i.e.,  $\Delta_{\mathbf{w}}^{(m)}(\mathbf{x}) \cong \Gamma(\mathbf{x})$ ). This can be also illustrated in terms of the spectral measures (Acerbi (2002)) as integrating by parts one gets

$$\begin{aligned} \mu_{\Gamma}(\mathbf{x}) &= 2 \int_0^1 F_{\mathbf{x}}^{(-2)}(\alpha)d\alpha = 2\alpha F_{\mathbf{x}}^{(-2)}(\alpha) \Big|_0^1 - 2 \int_0^1 \alpha F_{\mathbf{x}}^{(-1)}(\alpha)d\alpha \\ &= 2F_{\mathbf{x}}^{(-2)}(1) - 2 \int_0^1 \alpha F_{\mathbf{x}}^{(-1)}(\alpha)d\alpha = \int_0^1 2(1 - \alpha)F_{\mathbf{x}}^{(-1)}(\alpha)d\alpha \end{aligned}$$

which allows us to express the GMD safety measure by the risk aversion function  $\phi(\alpha) = 2(1 - \alpha)$  while formula (17) with the weights (18) defines a stepwise approximation to this function.

Again, the WCVaR measures may be considered the exact GMD measure applied to  $(m + 1)$ -point distributions approximating the original distribution of returns  $R_{\mathbf{x}}$ , thus providing a trapezoidal approximation to the original Lorenz dispersion space. In particular, for the  $(m + 1)$ -point distribution  $R_{\mathbf{x}}^{(\beta_1, \dots, \beta_m)}$

$$\mathbb{P}\{R_{\mathbf{x}}^{(\beta_1, \dots, \beta_m)} = \xi\} = \begin{cases} \beta_1, & \xi = a_1 \\ \beta_k - \beta_{k-1}, & \xi = a_k \quad \text{for } k = 1, \dots, m \\ 1 - \beta_m, & \xi = a_{m+1} \\ 0, & \text{otherwise} \end{cases}$$

such that  $a_1 \leq a_2 \leq \dots \leq a_{m+1}$ , the weighted conditional semideviation  $\Delta_{\mathbf{w}}^{(m)}(\mathbf{x}^{(\beta_1, \dots, \beta_m)})$  with weights (18) is equal to the Gini's mean difference  $\Gamma(\mathbf{x}^{(\beta_1, \dots, \beta_m)})$ . In general,  $\Delta_{\mathbf{w}}^{(m)}(\mathbf{x})$  is a lower approximation to  $\Gamma(\mathbf{x})$  ( $\Delta_{\mathbf{w}}^{(m)}(\mathbf{x}) \leq \Gamma(\mathbf{x})$ ) and thereby  $M_{\mathbf{w}}^{(m)}(\mathbf{x}) \geq \mu_{\Gamma}(\mathbf{x})$ .

It must be emphasized that despite being only an approximation to the Gini's mean difference, any WCVaR measure with weights defined by (18) is a well defined LP computable risk measure with guaranteed SSD consistency in the sense of Theorem 2. In other words, we are interested in the GMD approximation properties only for a reasonable weights definition. We will refer to the WCVaR measures

with weights defined by (18) as the Wide WCVaR as covering (spanning) a wide area of the quantile scale. The wide WCVaR measures need not to employ a very dense grid to provide a proper modeling of risk averse preferences. This allows us to build relatively small LP models with  $mT$  variables. In our computational analysis we have considered  $m = 3$  while testing three different patterns of the tolerance levels (see Table 1) corresponding to three types of preferences defined by the tolerance levels location.

### 3.3 Tail WCVaR measures

The Wide WCVaR measures, based on the approximation to the Gini's measure, contain the risk neutral term  $M_1(\mathbf{x}) = \mu(\mathbf{x})$  with the weight  $w_0 = 1 - \beta_m$ . This may cause the measure to pay too much attention to very low probable but very large returns. Actually, the measure can be more sensitive to large returns than the Gini's mean difference itself. We encountered such a situation in our computational analysis where in a few cases all the models based on the WCVaR approximation to GMD selected a single security portfolio with very high expectation caused by a very few but extremely high return realizations.

In order to overcome this flaw one may use the Tail WCVaR measures, built with an approximation to the tail GMD measures instead of the GMD itself. The tail Gini's measure (Ogryczak and Ruszczyński (2002a, 2002b)) defined for any  $\beta \in (0, 1]$  by averaging the vertical diameters  $h_p(\mathbf{x})$  within the tail interval  $p \leq \beta$  as:

$$\Gamma_\beta(\mathbf{x}) = \frac{2}{\beta^2} \int_0^\beta (\mu(\mathbf{x})\alpha - F_{\mathbf{x}}^{(-2)}(\alpha)) d\alpha. \quad (19)$$

A simple analysis of the absolute Lorenz curve (Ogryczak and Ruszczyński (2002a)) shows that, for any  $0 < \beta \leq 1$ , the tail Gini's measure  $\Gamma_\beta(\mathbf{x})$  is SSD safety consistent. One may notice that the corresponding safety measure  $\mu_{\Gamma_\beta}(\mathbf{x}) = \mu(\mathbf{x}) - \Gamma_\beta(\mathbf{x})$  can be expressed as

$$\mu_{\Gamma_\beta}(\mathbf{x}) = \mu(\mathbf{x}) - \frac{2}{\beta^2} \int_0^\beta (\mu(\mathbf{x})\alpha - F_{\mathbf{x}}^{(-2)}(\alpha)) d\alpha = \frac{2}{\beta^2} \int_0^\beta F_{\mathbf{x}}^{(-2)}(\alpha) d\alpha$$

which allows us to consider it as a second degree CVaR measure.

In the simplest case of equally probable  $T$  scenarios with  $p_t = 1/T$ , the tail Gini's measure for  $\beta = K/T$  may be expressed as the weighted conditional semideviation  $\Delta_{\mathbf{w}}^{(K)}(\mathbf{x})$  with tolerance levels  $\beta_k = k/T$  for  $k = 1, 2, \dots, K$  and properly defined weights (Ogryczak and Ruszczyński (2002a)). In a general case, we may resort to an approximation with the weighted CVaR measure based on some reasonably chosen grid  $\beta_k$ ,  $k = 1, \dots, m$  and weights  $w_k$  expressing the corresponding trapezoidal approximation of the integral in the formula (19). Exactly, for any  $0 < \beta \leq 1$ , while using the grid of  $m$  tolerance levels  $0 < \beta_1 < \dots < \beta_k < \dots < \beta_m = \beta$  one may define the weights:

$$w_k = \frac{(\beta_{k+1} - \beta_{k-1})\beta_k}{\beta^2}, \quad \text{for } k = 1, \dots, m-1, \quad \text{and} \quad w_m = \frac{(\beta_m - \beta_{m-1})\beta_m}{\beta^2} \quad (20)$$

where  $\beta_0 = 0$ . This results in the weighted sum  $\sum_{k=1}^m w_k \Delta_{\beta_k}(\mathbf{x})$  expressing the trapezoidal approximation to the tail Gini's measure (19). Note that  $\sum_{k=1}^m w_k = \beta_m^2 / \beta^2 = 1$  and thus we get a regular weighted conditional semideviation (13)  $\Delta_{\mathbf{w}}^{(m)}(\mathbf{x}) \cong \Gamma_\beta(\mathbf{x})$ . Further, weights (20) together with  $w_0 = 0$  generate a WCVaR measure (12) such that  $M_{\mathbf{w}}^{(m)}(\mathbf{x}) \cong \mu_{\Gamma_\beta}(\mathbf{x})$ . This can be also illustrated in terms of

the spectral measures (Acerbi (2002)) as integrating by parts one gets

$$\begin{aligned}\mu_{\Gamma_\beta}(\mathbf{x}) &= \frac{2}{\beta^2} \int_0^\beta F_{\mathbf{x}}^{(-2)}(\alpha) d\alpha = \frac{2}{\beta^2} \alpha F_{\mathbf{x}}^{(-2)}(\alpha) \Big|_0^\beta - \frac{2}{\beta^2} \int_0^\beta \alpha F_{\mathbf{x}}^{(-1)}(\alpha) d\alpha \\ &= \frac{2}{\beta} F_{\mathbf{x}}^{(-2)}(\beta) - \frac{2}{\beta^2} \int_0^\beta \alpha F_{\mathbf{x}}^{(-1)}(\alpha) d\alpha = \int_0^1 \frac{2(\beta - \alpha)^+}{\beta^2} F_{\mathbf{x}}^{(-1)}(\alpha) d\alpha\end{aligned}$$

allowing us to express the tail GMD safety measure by the risk aversion function  $\phi(\alpha) = 2(\beta - \alpha)^+ / \beta^2$  where  $(\cdot)^+$  denotes the nonnegative part of a number. Formula (17) with the weights (20) define a stepwise approximation to this function.

Again, we emphasize that despite being only an approximation to (19), any Tail WCVaR measure (e.g., a WCVaR measure with weights defined according to (20)) is a well defined LP computable measure with guaranteed SSD consistency in the sense of Theorem 2. They need not be built on a very dense grid to provide proper modeling of risk averse preferences. Actually, we are interested in a direct preference modeling with simple Tail WCVaR measures rather than strict approximation to the Tail GMD measure. In our computational analysis we have tested two Tail WCVaR models with  $m = 2$  and  $m = 3$ . Obviously, all the Tail WCVaR models measure are implemented as LP problems (16) but with  $w_0 = 0$ . Again, for a small value of  $m$  we get rather small LP models with  $mT$  variables.

### 3.4 Direct diversification enforcement

Since the seminal work of Markowitz (1952), the notion of investing in diversified portfolios is considered one of the most fundamental concepts of portfolio management. Diversification should be enforced by the mean/risk preference model. Indeed, in the original Markowitz model it was usually guaranteed by the standard deviation (variance) minimization. In general, it may happen that a single security or a low diversified portfolio is SSD dominating over all other (more diversified) portfolios, and the SSD consistent Markowitz-type models will select such an undiversified solution. Especially, the SSD consistent models based on the LP computable risk measures may fail to generate sufficiently diversified portfolios, although it also happens for the original Markowitz model (Mansini, Ogryczak and Speranza (2003b)). Therefore, additional restrictions may be posed on the feasible portfolios to guarantee the required diversification. The simplest way to enforce portfolio diversification is to limit the maximum share. This, however, allows us to form a portfolio with a few shares at the maximum level. A better modeling alternative would be to allow for a relatively large maximum share provided that the other shares are smaller. Such a rich diversification scheme may be introduced with the CVaR constructs applied to the right tail of the distribution of shares.

A natural generalization of the maximum share is the (right-tail) conditional mean defined as the mean within the specified tolerance level (amount) of the worst shares. One may simply define the conditional mean as the mean of the  $k$  largest shares. This can be formalized as follows. First, we introduce the ordering map  $\Theta : R^n \rightarrow R^n$  such that  $\Theta(\mathbf{x}) = (\theta_1(\mathbf{x}), \theta_2(\mathbf{x}), \dots, \theta_n(\mathbf{x}))$ , where  $\theta_1(\mathbf{x}) \geq \theta_2(\mathbf{x}) \geq \dots \geq \theta_n(\mathbf{x})$  and there exists a permutation  $\tau$  of set  $J$  such that  $\theta_j(\mathbf{x}) = x_{\tau(j)}$  for  $j = 1, \dots, n$ . The use of ordered outcome vectors  $\Theta(\mathbf{x})$  allows us to focus on distributions of shares impartially. Next, the linear cumulative map is applied to ordered vectors to get  $\bar{\Theta}(\mathbf{x}) = (\bar{\theta}_1(\mathbf{x}), \bar{\theta}_2(\mathbf{x}), \dots, \bar{\theta}_n(\mathbf{x}))$  defined

as

$$\bar{\theta}_k(\mathbf{x}) = \sum_{j=1}^k \theta_j(\mathbf{x}), \quad \text{for } k = 1, \dots, n.$$

The coefficients of vector  $\bar{\Theta}(\mathbf{x})$  express, respectively: the largest share, the total of the two largest shares, the total of the three largest shares, etc. Hence, the (worst)  $\frac{k}{n}$ -conditional mean share is given as  $\frac{1}{k} \bar{\theta}_k(\mathbf{x})$ , for  $k = 1, \dots, n$ .

Similar to the CVaR formulas, for a given vector  $\mathbf{x}$ , the value of  $\bar{\theta}_k(\mathbf{x})$  may be found by solving the linear program (Ogryczak and Tamir (2003)):

$$\bar{\theta}_k(\mathbf{x}) = \min \left\{ k s_k + \sum_{j=1}^n d_{kj}^s : d_{kj}^s \geq x_j - s_k, d_{kj}^s \geq 0 \quad \text{for } j = 1, \dots, n \right\},$$

where  $s_k$  is an unbounded variable (representing the  $k$ -th largest share at the optimum) and  $d_{kj}^s$  are additional nonnegative (deviational) variables. Hence, any model under consideration can easily be extended with direct diversification constraints specified as  $\bar{\theta}_k(\mathbf{x}) \leq c_k$  and implemented with linear inequalities:

$$k s_k + \sum_{j=1}^n d_{kj}^s \leq c_k \quad \text{and} \quad d_{kj}^s \geq x_j - s_k, d_{kj}^s \geq 0 \quad \text{for } j = 1, \dots, n. \quad (21)$$

## 4 Experimental analysis

### 4.1 Testing environment

The present section is devoted to the experimental analysis in a real framework of all the described LP models based on extensions of the CVaR measure. Models have been tested on a PC with a 500 MHz Pentium processor by using CPLEX 6.5 package. First we present the test problems, then the results of the in-sample analysis, both on the original models and on their modifications to enforce diversification, are described. Finally, the out-of-sample results are analyzed and the results obtained through the simulation of a "multiperiod-type" portfolio investment are presented.

Historical data are represented by weekly rates of return obtained by using stock prices from Milan Stock Exchange. The rates are computed as relative price variations. Dividends are not included. The data set consists of 157 securities quoted with continuity from 1994 to 1999. In the first years of this historical period the Italian Stock Exchange has shown alternate short periods of up and down trends while entering a positive growing trend at the end. This is shown in Figure 2, where the performance of the Milan Stock Exchange index MIB30 is depicted in the period (1994–1999).

A set of 13 instances has been created, each of which takes into account the complete set of securities over a different time period. For this reason, from now on, we will indifferently refer to them as instances or periods. In particular, each instance is based on two years realizations (about 104 weekly observations) as in-sample period and one year as out-of-sample. The choice of weekly periodicity is consistent with the objective of reducing estimation errors through an adequate number of observations (Simaan (1997)). Two consecutive instances differ from each other for a three months period, e.g. the first instance covers the two years 1994–1995 as in-sample period, while the second instance does not include the first three months of 1994 and does include the first three months of 1996. For each instance the Maximum Safety



Figure 2: The Milan Stock Exchange Index (MIB30) in the years 1994–1999 (weekly quotations).

Portfolio (MSP) has been obtained through the use of the various tested models. In this section we only summarize and comment the main figures out of the huge amount of computational results we obtained.

The model introduced by Young (1998), with safety measure the maximization of the worst realization (7), is identified as Minimax. The model based on the safety measure corresponding to the Gini's mean difference (10), i.e. the mean worse return, is referred simply as GMD. The CVaR model associated to a given tolerance level  $\beta$  is identified as  $CVaR(\beta)$ . We have tested the CVaR model for five different values of  $\beta$ , i.e.  $CVaR(0.05)$ ,  $CVaR(0.1)$ ,  $CVaR(0.25)$ ,  $CVaR(0.5)$  and  $CVaR(0.75)$ . All the CVaR and the weighted CVaR models have been formulated according to (16). The weighted CVaR models representing an approximation to the Gini's measure (the Wide CVaR models) have been tested with  $m = 3$  tolerance weights. Such models have been identified as  $WCVaR(AGD)$ ,  $WCVaR(AGS)$  and  $WCVaR(AGT)$ . The corresponding models weights are summarized in Table 1. Finally, we have also

Table 1: Wide Weighted CVaR models

Model	Tolerance levels	Weights			
		$w_0$	$w_1$	$w_2$	$w_3$
$WCVaR(AGD)$ :	Downside Approximation $\beta_1 = 0.1, \beta_2 = 0.25, \beta_3 = 0.5$	0.5	0.025	0.1	0.375
$WCVaR(AGS)$ :	Symmetric Approximation $\beta_1 = 0.25, \beta_2 = 0.5, \beta_3 = 0.75$	0.25	0.125	0.25	0.375
$WCVaR(AGT)$ :	Tails Approximation $\beta_1 = 0.1, \beta_2 = 0.5, \beta_3 = 0.9$	0.1	0.05	0.4	0.45

tested two Tail  $WCVaR$  models:

- Model  $WCVaR(TG2)$  with two tolerance levels  $\beta_1 = 0.1, \beta_2 = 0.25$  and weights  $w_1 = 0.4$  and

$$w_2 = 0.6.$$

- Model WCVaR(TG3) with three tolerance levels  $\beta_1 = 0.1$ ,  $\beta_2 = 0.25$ ,  $\beta_3 = 0.5$  and weights  $w_1 = 0.1$ ,  $w_2 = 0.4$  and  $w_3 = 0.5$ .

Since SSD consistent models based on the LP computable risk measures may fail to generate diversified enough portfolios we have added the following additional restrictions to guarantee the required diversification: any stock share cannot exceed 0.20, while any three shares cannot exceed 0.50 in total and any six shares cannot globally exceed 0.75 of the portfolio investment. This requires the following side constraints on the shares to be added:

$$\begin{aligned} x_j &\leq 0.2 && \text{for } j = 1, \dots, n \\ 3s_3 + \sum_{j=1}^n d_{3j}^s &\leq 0.5 \quad \text{and} \quad d_{3j}^s \geq x_j - s_3, \quad d_{3j}^s \geq 0 && \text{for } j = 1, \dots, n \\ 6s_6 + \sum_{j=1}^n d_{6j}^s &\leq 0.75 \quad \text{and} \quad d_{6j}^s \geq x_j - s_6, \quad d_{6j}^s \geq 0 && \text{for } j = 1, \dots, n \end{aligned} \quad (22)$$

For each model we have tested the corresponding version obtained by adding constraints (22).

## 4.2 In-sample analysis

In the following we present and comment the characteristics of the MSPs selected by the different models with and without the introduction of diversification enforcement. In Table 2, the complete computational results for model CVaR(0.1) are presented as an example of the type of information obtained by solving a single model over all the 13 periods. The table consists of a first part corresponding to the results for the models without diversification enforcement and a second one for the models with the diversification enforcement constraints (22). Each of the two parts of the table has five columns: the objective function value (obj.), the portfolio per cent mean return ( $z$ ), the portfolio diversification (div.) represented by the number of selected securities, the minimum and the maximum share within the portfolio, respectively. The average return is given on a weekly basis (a good yearly approximation can be obtained by multiplying the figures by 52). Notice that the introduction of diversification constraints may result in an unmodified optimal portfolio. This is the case for the portfolios selected in the instances 7 and 8. Similar results are shown in Table 3 for the model WCVaR(AGD).

To simplify results presentation, we have decided to focus our attention only on a subset of instances (periods). In Tables 4-6 we show the results obtained by the different models in the first, the sixth and the twelfth instance without and with diversification constraints. The tables have the same structure of Tables 2-3. Tables 4 and 5 show the large values obtained by the Wide WCVaR: the fact can be explained by the relevance given to high returns by these models which are much more sensitive to large returns than the Gini's mean difference itself. This also explains why, in some instances (see, for instance, Table 3), the Wide WCVaR models select only a one security portfolio characterized by a large expected return (about 94% per week) generated by very few realizations with dramatically high return. Moreover, it is worth noticing that in the Wide WCVaR models the mean return  $z$  value is larger than that of all the other models. This is also true in the twelfth period where the returns are dramatically lower than in other periods, thus reflecting the downwards trend of the whole market. From Tables 4-6 and the analogous results obtained for the other periods we observed that the MSPs mean return tends to increase over the years by reaching a pick in the first quarter of the year 1998 and then decreasing. This allows us to draw some conclusions on market trend. In general, the basic CVaR models are the most diversified.

Table 2: CVaR(0.1) model without and with diversification constraints: optimal portfolio characteristics in each of the 13 periods.

periods	without diversification					with diversification				
	obj. $10^{-2}$	$z$ %	div. #	shares min	max	obj. $10^{-2}$	$z$ %	div. #	shares min	max
1	-1.567	0.21	18	$1.77 \cdot 10^{-3}$	0.293	-1.638	0.17	20	$1.93 \cdot 10^{-3}$	0.200
2	-1.543	0.24	18	$2.09 \cdot 10^{-3}$	0.281	-1.604	0.21	20	$4.45 \cdot 10^{-4}$	0.200
3	-1.129	0.61	18	$3.33 \cdot 10^{-3}$	0.204	-1.137	0.52	20	$1.10 \cdot 10^{-3}$	0.192
4	-0.999	0.19	23	$1.68 \cdot 10^{-3}$	0.144	-0.999	0.19	23	$1.68 \cdot 10^{-3}$	0.144
5	-0.955	0.10	24	$6.24 \cdot 10^{-3}$	0.138	-0.955	0.10	24	$8.28 \cdot 10^{-3}$	0.138
6	-0.736	0.43	26	$1.18 \cdot 10^{-4}$	0.165	-0.736	0.43	26	$1.18 \cdot 10^{-4}$	0.132
7	-0.721	0.41	27	$1.48 \cdot 10^{-4}$	0.112	-0.721	0.41	27	$1.48 \cdot 10^{-4}$	0.112
8	-0.649	0.54	27	$1.53 \cdot 10^{-3}$	0.138	-0.649	0.54	27	$1.53 \cdot 10^{-3}$	0.138
9	-0.560	0.61	29	$7.64 \cdot 10^{-4}$	0.128	-0.560	0.61	29	$7.64 \cdot 10^{-4}$	0.128
10	-0.549	1.01	24	$2.07 \cdot 10^{-3}$	0.121	-0.549	1.01	24	$2.07 \cdot 10^{-3}$	0.121
11	-1.401	0.99	19	$7.81 \cdot 10^{-5}$	0.245	-1.412	0.98	20	$1.05 \cdot 10^{-3}$	0.200
12	-2.188	0.91	18	$1.11 \cdot 10^{-4}$	0.210	-2.188	0.91	18	$1.11 \cdot 10^{-4}$	0.210
13	-2.476	0.90	17	$4.49 \cdot 10^{-3}$	0.223	-2.478	0.87	18	$5.30 \cdot 10^{-4}$	0.200

Moreover, in some of these models (see CVaR(0.05) and CVaR(0.1)) the introduction of constraints (22) results in no further diversification (see Table 5).

In the following the main figures obtained for all the selected portfolios are summarized. Table 7 shows, for all the models over all the periods, the diversification of the optimal portfolios (MSPs). For instance, the number of selected securities for the Minimax model varies, out of the 13 solved instances, from 6 to 29 securities, while with the introduction of diversification constraints the corresponding range becomes 12–29. Similar considerations can be made by analyzing portfolios composition in terms of minimum and maximum portfolio shares. On average, the Wide WCVaR provide the ranges with the lowest upper limits and result in extremely low diversified portfolios (or rather undiversified portfolios as the lower limit can be equal to 1). These models seem to require the use of an additional technique to guarantee enough diversification. Also the Minimax model may generate some low diversified portfolios (6 securities). The other models have always selected more than 10 securities. Nevertheless, in many cases they generated portfolios with very large shares (exceeding 30%) of particular securities. Thus, for all the LP computable models under consideration we may recommend a support of some direct technique for diversification enforcement. One may notice that the application of the CVaR based diversification enforcement constraints (22) has resulted in portfolios always containing at least 10 securities.

### 4.3 Out-of-sample analysis

In this section the behavior of all the MSPs is examined in the twelve months following the date of each portfolio selection. To describe out-of-sample results we have used the following nine ex-post parameters: the minimum, the average, the maximum and the median portfolio return ( $r_{min}$ ,  $r_{av}$ ,  $r_{max}$  and  $r_{med}$ , respectively); the standard deviation (std) and the semi-standard deviation (s-std); the mean absolute deviation (MAD) and the mean downside semideviation (s-MAD); the maximum downside deviation (D-DEV). Such performance criteria have been computed for all the models over all the periods and can be used to compare the out-of-sample behavior of the maximum safety portfolios selected by the different models. The minimum, average, maximum and median ex-post portfolio returns are expressed

Table 3: WCVaR(AGD) without and with diversification constraints: optimal portfolio characteristics in each of the 13 periods.

periods	without diversification					with diversification				
	obj. $10^{-2}$	$z$ %	div. #	shares		obj. $10^{-2}$	$z$ %	div. #	shares	
				min	max				min	max
1	40.577	94.26	1	1	1	8.422	19.52	12	$1.49 \cdot 10^{-2}$	0.200
2	40.464	93.88	1	1	1	8.462	19.57	11	$1.56 \cdot 10^{-2}$	0.200
3	42.558	95.04	1	1	1	8.842	19.67	12	$4.34 \cdot 10^{-3}$	0.200
4	42.467	95.21	1	1	1	8.804	19.64	14	$3.63 \cdot 10^{-3}$	0.200
5	42.991	95.46	1	1	1	8.881	19.55	19	$8.29 \cdot 10^{-4}$	0.200
6	45.953	97.16	1	1	1	9.535	20.06	15	$8.77 \cdot 10^{-3}$	0.200
7	0.270	1.11	15	$1.18 \cdot 10^{-3}$	0.204	0.270	1.10	16	$8.12 \cdot 10^{-4}$	0.200
8	0.313	1.19	17	$7.29 \cdot 10^{-3}$	0.198	0.313	1.19	17	$7.29 \cdot 10^{-3}$	0.198
9	0.332	1.00	21	$4.53 \cdot 10^{-3}$	0.106	0.332	1.00	21	$4.53 \cdot 10^{-3}$	0.106
10	0.673	1.57	21	$2.49 \cdot 10^{-3}$	0.148	0.673	1.57	21	$2.49 \cdot 10^{-3}$	0.148
11	0.405	1.51	16	$6.30 \cdot 10^{-3}$	0.168	0.405	1.51	16	$6.30 \cdot 10^{-3}$	0.168
12	0.182	1.51	17	$2.71 \cdot 10^{-3}$	0.179	0.182	1.51	17	$2.71 \cdot 10^{-3}$	0.179
13	0.219	1.87	14	$1.32 \cdot 10^{-2}$	0.237	0.217	1.84	15	$8.64 \cdot 10^{-3}$	0.200

Table 4: Period 1 - Maximum Safety Portfolios: optimal portfolio characteristics.

models	without diversification					with diversification				
	obj. $10^{-2}$	$z$ %	div. #	shares		obj. $10^{-2}$	$z$ %	div. #	shares	
				min	max				min	max
Minimax	-1.812	0.06	14	$1.40 \cdot 10^{-3}$	0.193	-1.823	0.12	16	$3.98 \cdot 10^{-4}$	0.194
CVaR(0.05)	-1.767	0.15	14	$5.14 \cdot 10^{-3}$	0.214	-1.789	0.13	16	$3.06 \cdot 10^{-3}$	0.187
CVaR(0.1)	-1.567	0.21	18	$1.77 \cdot 10^{-3}$	0.293	-1.638	0.17	20	$1.93 \cdot 10^{-3}$	0.200
CVaR(0.25)	-1.155	0.29	11	$1.57 \cdot 10^{-3}$	0.316	-1.199	0.29	18	$5.13 \cdot 10^{-3}$	0.200
CVaR(0.5)	-0.597	0.39	17	$3.31 \cdot 10^{-5}$	0.288	-0.611	0.40	19	$4.87 \cdot 10^{-5}$	0.200
CVaR(0.75)	-0.055	0.62	17	$1.12 \cdot 10^{-4}$	0.179	-0.056	0.61	18	$1.18 \cdot 10^{-4}$	0.172
GMD	-0.313	0.60	17	$2.60 \cdot 10^{-4}$	0.246	-0.318	0.61	18	$7.90 \cdot 10^{-4}$	0.200
WCVaR(AGD)	40.577	94.26	1	1	1	8.422	19.52	12	$1.49 \cdot 10^{-2}$	0.200
WCVaR(AGS)	16.147	94.26	1	1	1	3.532	19.48	13	$1.19 \cdot 10^{-2}$	0.200
WCVaR(AGT)	1.603	94.26	1	1	1	0.632	19.47	13	$2.74 \cdot 10^{-3}$	0.200
WCVaR(TG2)	-1.393	0.21	16	$1.34 \cdot 10^{-3}$	0.343	-1.455	0.16	18	$2.63 \cdot 10^{-3}$	0.200
WCVaR(TG3)	-0.986	0.31	15	$8.99 \cdot 10^{-5}$	0.304	-1.025	0.36	18	$2.60 \cdot 10^{-5}$	0.200

on a yearly basis. All the dispersion measures (std, s-std, MAD, s-MAD and D-DEV) have been computed with respect to the target return  $\mu_0$  (which is zero for the MSP) to make them directly comparable in the different models.

In Tables 8 and 9 we present the average value of each criterion, over the thirteen periods, for the various models in the cases without and with diversification enforcement, respectively. One may notice extremely high average returns of the Wide WCVaR models (without diversification enforcement). These performances are produced by single security portfolios with very high returns. In general, the models are too risky as demonstrated by all the dispersion measures. When we consider the models with diversification enforcement (Table 9), the Wide WCVaR models are still characterized by the highest average returns and the largest dispersion parameters but the differences from the other models are not very large. One may notice that the GMD model, which is the computationally most complex, may be easily outperformed (in terms of average returns and dispersion) by the simpler Tail WCVaR models or even by the CVaR(0.5) model.

Table 5: Period 6 - Maximum Safety Portfolios: optimal portfolio characteristics.

models	without diversification					with diversification				
	obj. $10^{-2}$	$z$ %	div. #	shares		obj. $10^{-2}$	$z$ %	div. #	shares	
				min	max				min	max
Minimax	-0.745	0.45	27	$2.31 \cdot 10^{-4}$		-0.745	0.45	27	$2.31 \cdot 10^{-4}$	
CVaR(0.05)	-0.745	0.45	27	$2.31 \cdot 10^{-4}$		-0.745	0.45	27	$2.31 \cdot 10^{-4}$	
CVaR(0.1)	-0.736	0.43	26	$1.18 \cdot 10^{-4}$		-0.736	0.43	26	$1.18 \cdot 10^{-4}$	
CVaR(0.25)	-0.613	0.44	22	$1.27 \cdot 10^{-5}$		-0.613	0.44	22	$1.27 \cdot 10^{-5}$	
CVaR(0.5)	-0.246	2.64	26	$3.30 \cdot 10^{-3}$		-0.246	2.64	26	$3.30 \cdot 10^{-3}$	
CVaR(0.75)	0.262	7.33	21	$3.72 \cdot 10^{-4}$		0.260	7.01	21	$6.88 \cdot 10^{-3}$	
GMD	0.060	9.12	20	$4.30 \cdot 10^{-3}$		0.060	9.12	20	$4.11 \cdot 10^{-3}$	
WCVaR(AGD)	45.953	97.16	1	1	1	9.535	20.06	15	$8.77 \cdot 10^{-3}$	
WCVaR(AGS)	21.599	97.16	1	1	1	4.651	20.01	16	$6.17 \cdot 10^{-3}$	
WCVaR(AGT)	7.504	97.16	1	1	1	1.868	20.07	14	$2.41 \cdot 10^{-5}$	
WCVaR(TG2)	-0.693	0.44	27	$1.19 \cdot 10^{-4}$		-0.693	0.44	27	$1.19 \cdot 10^{-4}$	
WCVaR(TG3)	-0.514	1.61	26	$1.47 \cdot 10^{-3}$		-0.514	1.61	26	$1.47 \cdot 10^{-3}$	

Table 6: Period 12 - Maximum Safety Portfolios: optimal portfolio characteristics.

models	without diversification					with diversification				
	obj. $10^{-2}$	$z$ %	div. #	shares		obj. $10^{-2}$	$z$ %	div. #	shares	
				min	max				min	max
Minimax	-2.848	0.99	13	$3.77 \cdot 10^{-3}$		-2.927	0.99	14	$1.41 \cdot 10^{-3}$	
CVaR(0.05)	-2.486	0.88	14	$1.09 \cdot 10^{-3}$		-2.488	0.88	15	$1.51 \cdot 10^{-3}$	
CVaR(0.1)	-2.188	0.91	18	$1.11 \cdot 10^{-4}$		-2.188	0.90	18	$9.13 \cdot 10^{-4}$	
CVaR(0.25)	-1.529	0.94	20	$4.65 \cdot 10^{-3}$		-1.529	0.94	20	$4.65 \cdot 10^{-3}$	
CVaR(0.5)	-0.608	1.05	20	$7.87 \cdot 10^{-4}$		-0.608	1.05	20	$7.87 \cdot 10^{-4}$	
CVaR(0.75)	0.204	1.35	14	$4.30 \cdot 10^{-4}$		0.203	1.35	14	$3.99 \cdot 10^{-3}$	
GMD	-0.186	1.35	20	$1.01 \cdot 10^{-2}$		-0.186	1.35	20	$1.01 \cdot 10^{-2}$	
WCVaR(AGD)	0.182	1.51	17	$2.71 \cdot 10^{-3}$		0.182	1.51	17	$2.71 \cdot 10^{-3}$	
WCVaR(AGS)	-0.028	1.39	18	$6.10 \cdot 10^{-3}$		-0.028	1.39	18	$6.10 \cdot 10^{-3}$	
WCVaR(AGT)	0.029	1.41	20	$3.36 \cdot 10^{-3}$		0.029	1.41	20	$3.36 \cdot 10^{-3}$	
WCVaR(TG2)	-1.882	0.99	18	$1.25 \cdot 10^{-3}$		-1.882	0.99	18	$1.25 \cdot 10^{-3}$	
WCVaR(TG3)	-1.270	0.99	23	$1.47 \cdot 10^{-4}$		-1.270	0.99	23	$1.47 \cdot 10^{-4}$	

We have also analyzed each model performance with respect to a long-run portfolio management. Each of the portfolios selected by a specific model in the 13 instances has been evaluated ex-post in the three months period following the date of selection. Tables 10 and 11 provide the single period returns (each column corresponds to a different period from 1 to 13) for each model without and with the diversification enforcement, respectively. It is worth noticing that single period ex-post returns quite perfectly represent the upward and downward movements of the market. For instance, the negative results showed by all the models in the periods 3 and 4 are mainly due to the negative trend of the market in 1996, especially in the period from April to July and then again in October. However, we noticed that in such periods some models (such as CVaR(0.1) and CVaR(0.25)) find portfolios with a better performance with respect to the market index MIB30. Similarly, for the periods 10-11 and partially 12 which suffered the negative fall of the market in the second quarter of the 1998 and then again in August. Finally, the high returns in period 9 can be partially interpreted as a consequence of the positive trend of the market at the beginning of the 1998 with a high positive jump of the MIB30 index performances in March (see

Table 7: Diversification of the optimal portfolios (MSPs).

models	MSP without diversification enforcement	MSP with diversification enforcement
Minimax	6–29	12–29
CVaR(0.05)	14–29	15–29
CVaR(0.1)	17–29	18–29
CVaR(0.25)	11–30	18–30
CVaR(0.5)	16–29	18–27
CVaR(0.75)	12–23	13–22
GMD	12–26	16–26
WCVaR(AGD)	1–21	11–21
WCVaR(AGS)	1–23	13–21
WCVaR(AGT)	1–25	11–25
WCVaR(TG2)	15–30	17–30
WCVaR(TG3)	15–29	16–29

Table 8: Out-of-sample statistics for MSPs: average values over the 13 periods.

models	Without diversification enforcement								
	$r_{min}$	$r_{av}$	$r_{med}$	$r_{max}$	std	s-std	MAD	s-MAD	D-DEV
Minimax	-54.36	30.05	17.57	337.19	0.0614	0.0270	0.0493	0.0148	0.0749
CVaR(0.05)	-52.75	28.77	10.66	332.93	0.0596	0.0264	0.0477	0.0145	0.0720
CVaR(0.1)	-52.43	30.57	12.24	347.98	0.0611	0.0263	0.0486	0.0141	0.0713
CVaR(0.25)	-49.68	34.27	26.58	307.67	0.0591	0.0237	0.0454	0.0110	0.0718
CVaR(0.5)	-53.10	31.12	32.34	282.05	0.0603	0.0267	0.0474	0.0128	0.0788
CVaR(0.75)	-59.86	27.80	29.34	381.36	0.0668	0.0325	0.0512	0.0161	0.0945
GMD	-58.64	27.85	26.21	315.79	0.0639	0.0304	0.0495	0.0151	0.0870
WCVaR(AGD)	-72.84	92.27	48.52	3625.59	0.1153	0.0427	0.0913	0.0218	0.1148
WCVaR(AGS)	-71.69	94.00	47.57	3634.52	0.1152	0.0415	0.0907	0.0209	0.1136
WCVaR(AGT)	-72.17	94.15	49.47	3631.79	0.1151	0.0418	0.0908	0.0209	0.1146
WCVaR(TG2)	-52.14	30.52	27.86	378.51	0.0614	0.0259	0.0473	0.0134	0.0719
WCVaR(TG3)	-49.50	33.09	25.23	321.26	0.0602	0.0243	0.0458	0.0116	0.0719

Figure 2). In such period many models find higher returns with respect to the MIB30 index.

Further, we cumulated the returns over the horizon up to 13 periods (39 months) to better analyze each model achievements. The figures shown in Table 12 are the cumulative returns of the portfolios selected by each model in the case without diversification enforcement. Table 13 provides the same results for the case with diversification enforcement. Each column of these tables refers to a period and provides the cumulative returns of the portfolios selected over the preceding periods. For a better understanding of these figures let us consider the first line of Table 12 which refers to the model Minimax. Each of the 13 portfolios selected by the Minimax model in the 13 instances has been evaluated ex-post in the three months investment period following the date of its selection. Let us define as  $r_1, r_2, \dots, r_{13}$  the ex-post returns of these 13 portfolios (their values are shown in the first line of Table 10). Then, the first column of Table 12 gives the ex-post return (after 3 months) of the first portfolio selected, i.e.  $r_1$  (notice that such value is identical in Tables 12 and 10). The second column of Table 12 gives the cumulative return of the portfolio selected in the first period and then modified after three months with the portfolio selected in the second period: the value is computed as  $(1 + r_1)(1 + r_2) - 1$ . Similarly, for all the other columns of the table. These results have been computed to simulate a multi-period setting where, at no transaction cost, the portfolio changes over time. Rates are expressed on a yearly basis.

Table 9: Out-of-sample statistics for MSPs: average values over the 13 periods.

models	With diversification enforcement								
	$r_{min}$	$r_{av}$	$r_{med}$	$r_{max}$	std	s-std	MAD	s-MAD	D-DEV
Minimax	-53.80	31.38	25.49	337.79	0.0607	0.0263	0.0486	0.0139	0.0735
CVaR(0.05)	-53.27	28.73	13.78	329.62	0.0592	0.0264	0.0473	0.0143	0.0731
CVaR(0.1)	-52.17	29.88	11.94	334.94	0.0596	0.0259	0.0474	0.0138	0.0712
CVaR(0.25)	-46.93	33.00	26.57	302.67	0.0581	0.0232	0.0444	0.0109	0.0678
CVaR(0.5)	-53.50	30.49	32.33	289.09	0.0607	0.0268	0.0470	0.0129	0.0793
CVaR(0.75)	-60.23	28.56	29.34	384.64	0.0682	0.0333	0.0522	0.0163	0.0948
GMD	-59.19	27.50	27.78	323.99	0.0646	0.0308	0.0497	0.0153	0.0877
WCVaR(AGD)	-63.43	35.21	38.88	439.76	0.0742	0.0337	0.0582	0.0171	0.0947
WCVaR(AGS)	-62.91	36.75	36.00	426.09	0.0737	0.0327	0.0574	0.0162	0.0944
WCVaR(AGT)	-62.67	36.96	38.21	426.26	0.0735	0.0328	0.0575	0.0162	0.0945
WCVaR(TG2)	-52.04	30.19	27.86	366.65	0.0608	0.0257	0.0466	0.0132	0.0719
WCVaR(TG3)	-49.36	31.62	25.44	322.26	0.0600	0.0246	0.0452	0.0119	0.0718

Table 10: Out-of-sample results for MSPs: single period returns.

models	Without diversification enforcement												
	1	2	3	4	5	6	7	8	9	10	11	12	13
Minimax	54.09	13.15	-16.33	-2.41	62.07	33.31	39.77	85.43	348.50	-24.86	-60.31	0.99	10.33
CVaR(0.05)	63.87	11.71	-16.33	-2.41	62.07	33.31	39.77	85.43	348.50	-24.86	-49.44	-2.84	9.51
CVaR(0.1)	64.54	11.80	-10.97	-4.88	65.87	27.61	39.23	78.16	352.58	-25.94	-48.17	-7.46	39.95
CVaR(0.25)	73.05	4.27	-5.02	-0.40	80.39	38.78	20.16	71.22	392.11	10.28	-54.66	31.95	34.90
CVaR(0.5)	96.96	20.96	-21.01	-4.74	52.95	41.05	10.14	58.21	434.77	7.51	-55.21	39.74	58.90
CVaR(0.75)	148.85	32.11	-13.21	-19.60	58.17	48.33	11.85	37.56	519.56	2.30	-58.02	87.45	4.59
GMD	128.21	29.54	-14.26	-18.64	60.09	77.01	9.75	36.80	431.47	10.76	-53.08	47.40	10.33
WCVaR(AGD)	322.73	1221.48	-13.19	79.75	245.78	186.20	12.80	39.12	442.15	3.11	-52.42	44.00	10.33
WCVaR(AGS)	322.73	1221.48	-13.19	79.75	245.78	186.20	7.30	40.06	440.93	0.26	-55.78	57.47	10.46
WCVaR(AGT)	322.73	1221.48	-13.19	79.75	245.78	186.20	12.76	29.69	453.65	8.57	-54.52	40.90	17.69
WCVaR(TG2)	53.17	1.84	-11.14	-1.79	81.30	30.09	28.27	71.23	385.45	-17.94	-47.15	10.67	140.90
WCVaR(TG3)	78.91	12.06	-11.77	-6.49	77.17	42.13	21.41	52.27	404.90	7.78	-53.40	29.80	66.31

Table 12 shows extremely high cumulated returns of the Wide WCVaR models (without diversification enforcement). These performances are due to the single security portfolios selected in the first 6 periods which resulted in dramatically high returns. Actually, when ignoring these 6 periods and focusing on the remaining horizon of the last 21 months, the cumulative returns of the Wide WCVaR models considerably shrink as it is evident when comparing the first part of Table 14 with the last seven columns of Table 12. Table 14 shows the ex-post cumulated returns over the last 21 months (7 periods) both for the case without and with diversification enforcement. Moreover, as in column (7–13) of the first part of Table 14, it can be noticed that the Wide WCVaR models perform much worse than all the other models except for the Minimax and the extremal CVaR models ( $\beta = 0.05$  or  $\beta = 0.1$ ). Note that both the Tail WCVaR models and the CVaR(0.5) here have the best cumulative performances.

When the models with diversification enforcement are considered (see Table 13), the Wide WCVaR models are still characterized by the highest cumulative returns but the differences from the other models are not very large. When ignoring the first 6 periods and focusing on the last 21 months (second part of Table 14), one may see again the Wide WCVaR models performing much worse than all the other models except for the Minimax and the extremal CVaR models. It is interesting to notice that, except for

Table 11: Out-of-sample results for MSPs: single period returns.

models	With diversification enforcement												
	1	2	3	4	5	6	7	8	9	10	11	12	13
Minimax	67.81	13.26	-26.42	-2.25	62.07	33.31	39.77	85.43	348.50	-24.86	-61.65	-11.03	-0.52
CVaR(0.05)	57.11	16.46	-26.42	-2.25	62.07	33.31	39.77	85.43	348.50	-24.86	-54.08	-2.95	6.28
CVaR(0.1)	48.16	11.85	-15.53	-4.88	65.87	27.61	39.23	78.16	352.58	-25.94	-49.31	-7.63	25.82
CVaR(0.25)	71.10	4.17	-12.74	-0.40	80.39	38.78	20.16	71.22	392.11	10.28	-54.02	31.95	34.90
CVaR(0.5)	100.55	15.53	-21.94	-5.39	52.93	41.05	10.14	58.21	434.77	7.51	-55.21	39.74	58.90
CVaR(0.75)	145.52	29.29	-13.17	-21.09	57.85	46.87	16.47	37.56	519.56	2.30	-59.99	89.12	8.76
GMD	127.84	20.23	-18.40	-20.27	59.58	76.91	10.92	36.80	431.47	10.76	-52.45	47.40	10.39
WCVaR(AGD)	184.78	107.76	-23.10	-6.83	84.90	85.39	12.68	39.12	442.15	3.11	-52.42	44.00	9.76
WCVaR(AGS)	170.49	100.48	-21.68	-6.28	86.56	93.73	10.06	40.06	440.93	0.26	-55.32	57.47	11.57
WCVaR(AGT)	173.36	101.80	-21.99	-7.87	95.32	98.48	13.00	29.69	453.65	8.57	-54.52	40.90	17.69
WCVaR(TG2)	50.35	7.30	-14.16	-1.79	81.30	30.09	28.27	71.23	385.45	-17.94	-47.30	10.67	129.28
WCVaR(TG3)	81.68	7.66	-15.61	-6.41	77.17	42.13	21.41	52.27	404.90	7.78	-56.11	29.80	66.31

Table 12: Out-of-sample results for MSPs: cumulative returns.

models	Without diversification enforcement												
	3 m.	6 m.	9 m.	12 m.	15 m.	18 m.	21 m.	24 m.	27 m.	30 m.	33 m.	36 m.	39 m.
Minimax	54.09	32.04	13.41	9.23	18.20	20.59	23.16	29.63	48.80	38.97	24.00	21.90	20.97
CVaR(0.05)	63.87	35.30	15.27	10.57	19.36	21.58	24.03	30.42	49.60	39.65	27.33	24.49	23.27
CVaR(0.1)	64.54	35.63	17.87	11.72	20.91	22.00	24.33	30.04	49.37	39.25	27.28	23.95	25.11
CVaR(0.25)	73.05	34.33	19.67	14.30	25.23	27.39	26.33	31.22	51.98	47.19	32.25	32.22	32.43
CVaR(0.5)	96.96	54.35	23.46	15.71	22.35	25.29	23.00	26.93	48.93	44.15	29.62	30.43	32.43
CVaR(0.75)	148.85	81.32	41.84	23.07	29.40	32.38	29.23	30.24	54.89	48.60	32.46	36.35	33.60
GMD	128.21	71.94	36.35	19.84	26.98	34.21	30.41	31.19	53.25	48.36	33.61	34.71	32.66
WCVaR(AGD)	322.73	647.41	264.67	205.56	213.21	208.53	167.22	146.28	168.85	144.28	110.52	103.96	94.55
WCVaR(AGS)	322.73	647.41	264.67	205.56	213.21	208.53	165.32	144.96	167.49	142.49	107.74	102.99	93.71
WCVaR(AGT)	322.73	647.41	264.67	205.56	213.21	208.53	167.21	144.12	167.38	144.33	109.71	102.87	94.55
WCVaR(TG2)	53.17	24.90	11.50	8.02	19.80	21.46	22.41	27.66	48.08	39.59	27.80	26.27	32.71
WCVaR(TG3)	78.91	41.59	20.94	13.41	23.99	26.85	26.05	29.07	50.19	45.29	31.02	30.92	33.35

the Minimax and the extremal CVaR models, all the other models resulted in similar cumulative return over the entire horizon of 39 months with (annual) rate of return exceeding 30%. Also the GMD model is outperformed by simple Tail WCVaR models and the CVaR models for larger tolerance levels.

To better capture the models behavior over small periods with possibly different market trends we have analyzed the ex-post cumulative returns over subperiods of length 4, that is we computed the cumulative returns over the periods 7–10, 8–11, 9–12 and 10–13. In Table 15 we have shown the minimum, the average and the maximum cumulative return ( $rc_{min}$ ,  $rc_{av}$ , and  $rc_{max}$ , respectively) for each model over these subperiods. Note that during the periods with negative market trend, as during subperiod 10–13, all the models have negative average cumulative returns. However, GMD and CVaR(0.5) and CVaR(0.25) show the best, although negative, average performance. Moreover, when the maximum cumulative return is considered, the three best models are GMD, CVaR(0.25) and WCVaR(TG3), respectively. On the contrary, in positive market trend periods such as in the subperiod 9-12 (corresponding to the first part of the year 1998), the CVaR(0.75) has the largest average and maximum cumulative return while the largest minimum cumulative return is obtained by GMD.

Finally, to show how consistently the composition of the portfolios selected by the same model

Table 13: Out-of-sample results for MSPs: cumulative returns.

models	With diversification enforcement												
	3 m.	6 m.	9 m.	12 m.	15 m.	18 m.	21 m.	24 m.	27 m.	30 m.	33 m.	36 m.	39 m.
Minimax	67.81	37.87	11.83	8.13	17.25	19.78	22.45	28.97	48.13	38.41	23.16	19.87	18.16
CVaR(0.05)	57.11	35.26	10.42	7.11	16.36	19.03	21.79	28.36	47.50	37.88	24.77	22.18	20.88
CVaR(0.1)	48.16	28.73	11.86	7.42	17.17	18.85	21.57	27.52	46.79	37.08	25.23	22.09	22.38
CVaR(0.25)	71.10	33.50	15.86	11.56	22.82	25.34	24.59	29.64	50.35	45.76	31.25	31.31	31.58
CVaR(0.5)	100.55	52.22	21.84	14.37	21.21	24.32	22.18	26.19	48.16	43.48	29.07	29.93	31.95
CVaR(0.75)	145.52	78.16	40.21	21.44	27.98	30.95	28.78	29.84	54.46	48.23	31.59	35.63	33.35
GMD	127.84	65.51	30.75	15.54	23.25	30.90	27.84	28.93	50.90	46.31	32.10	33.31	31.39
WCVaR(AGD)	184.78	143.24	65.70	43.49	50.95	56.21	49.09	47.81	70.77	62.37	45.22	45.12	42.03
WCVaR(AGS)	170.49	132.87	61.94	41.25	49.33	55.95	48.38	47.31	70.22	61.44	43.65	44.75	41.88
WCVaR(AGT)	173.36	134.87	62.66	41.11	50.59	57.68	50.35	47.60	70.96	63.37	45.44	45.06	42.74
WCVaR(TG2)	50.35	27.01	11.46	7.99	19.78	21.44	22.39	27.64	48.06	39.58	27.75	26.23	32.16
WCVaR(TG3)	81.68	39.85	18.18	11.49	22.31	25.41	24.83	27.97	49.05	44.30	29.50	29.52	32.04

Table 14: Out-of-sample computational results: cumulative returns over the latest 21 months (7 periods).

models	periods	Without diversification							With diversification						
		7	7-8	7-9	7-10	7-11	7-12	7-13	7	7-8	7-9	7-10	7-11	7-12	7-13
Minimax		39.77	60.99	115.24	71.91	28.23	23.22	22.83	39.77	60.99	115.24	71.91	27.35	19.96	16.79
CVaR(0.05)		39.77	60.99	115.24	71.91	34.59	27.47	24.74	39.77	60.99	115.24	71.91	32.02	25.42	22.49
CVaR(0.1)		39.23	57.50	112.92	69.81	33.93	25.93	27.84	39.23	57.50	112.92	69.81	33.34	25.42	25.48
CVaR(0.25)		20.16	43.44	106.15	8.28	38.32	37.24	36.90	20.16	43.44	106.15	8.28	38.70	37.55	37.17
CVaR(0.5)		10.14	32.00	100.87	77.91	35.02	35.79	38.88	10.14	32.00	100.87	77.91	35.02	35.79	38.88
CVaR(0.75)		11.85	24.04	102.30	76.72	32.56	40.44	34.65	16.47	26.58	104.88	78.51	32.37	40.48	35.43
GMD		9.75	22.53	91.36	72.42	32.90	35.21	31.34	10.92	23.18	92.00	72.88	33.55	35.76	31.81
WCVaR(AGD)		12.80	25.27	95.24	72.10	33.08	34.84	29.74	12.68	25.20	95.17	72.05	33.05	34.81	30.91
WCVaR(AGS)		7.30	22.59	92.48	68.97	29.23	33.56	29.98	10.06	24.15	94.01	70.04	30.16	34.36	30.84
WCVaR(AGT)		12.76	20.93	92.24	72.19	31.94	33.39	31.03	13.00	21.06	92.37	72.28	32.00	33.44	31.07
WCVaR(TG2)		28.27	48.21	109.51	71.99	35.84	31.28	43.17	28.27	48.21	109.51	71.99	35.76	31.21	42.10
WCVaR(TG3)		21.41	35.97	100.98	78.10	36.20	35.12	39.19	21.41	35.97	100.98	78.10	34.59	33.78	38.00

over the different periods may change, we have reported, as an example, Table 16 which provides the portfolios composition changes from one period to the other for the portfolios selected by the different models in the case without diversification enforcement. For instance, the second line of Table 16 refers to model CVaR(0.05) and can be interpreted as follows. The first column gives the number of securities selected by this model in the first period (in this case 14 securities). The second column says that, with respect to the previous portfolio, the one selected in the second period contains 2 new securities and no securities have been eliminated from those already selected. Similarly for the other models.

#### 4.4 Models behavior in a strong downward trend period: the years 2000-2002

The Markowitz type models, used without any additional forecasting procedure applied prior to portfolio selection process itself, do not recognize any market trends and therefore they are generally not appropriate tools for investment situations with a long lasting market trend. Nevertheless, due to commonly observed negative trends during recent years, both researchers and practitioners become more interested

Table 15: Cumulative return statistics in subperiods of length 4: case without diversification.

models	7-10			8-11			9-12			10-13		
	$r_{Cmin}$	$r_{Cav}$	$r_{Cmax}$									
Minimax	39.77	71.98	115.24	24.42	93.88	188.38	3.55	110.99	348.50	-46.32	-33.69	-24.86
CVaR(0.05)	39.77	71.98	115.24	30.15	95.31	188.38	10.70	114.35	348.50	-41.26	-29.46	-22.76
CVaR(0.1)	39.23	69.87	112.92	31.90	92.20	183.96	11.93	116.40	352.58	-38.73	-27.90	-18.73
CVaR(0.25)	8.28	44.51	106.15	43.77	101.49	190.28	33.07	148.21	392.11	-28.79	-8.21	10.28
CVaR(0.5)	10.14	55.23	100.87	42.07	97.63	190.87	34.39	161.67	434.77	-30.61	-8.27	7.51
CVaR(0.75)	16.47	56.61	104.88	36.67	90.73	191.93	33.75	188.27	519.56	-36.02	-11.41	2.30
GMD	10.92	49.75	92.00	36.80	84.56	169.64	37.94	163.64	431.47	-27.42	-7.01	10.76
WCVaR(AGD)	12.68	51.28	95.17	38.69	85.57	174.63	35.75	163.56	442.15	-29.96	-10.83	3.11
WCVaR(AGS)	10.06	49.57	94.01	35.73	84.87	175.25	31.87	161.36	440.93	-33.07	-12.24	0.26
WCVaR(AGT)	13.00	49.68	92.37	29.69	81.21	167.96	36.93	168.97	453.65	-29.73	-9.19	8.57
WCVaR(TG2)	28.27	64.50	109.51	37.70	94.87	188.31	23.46	133.65	385.45	-34.24	-17.60	2.35
WCVaR(TG3)	21.41	59.12	100.98	38.10	90.32	177.28	31.27	150.53	404.90	-31.22	-9.27	7.78

Table 16: Out-of-sample computational results: changes in portfolios composition.

models	Without diversification enforcement												
	3 m.	6 m.	9 m.	12 m.	15 m.	18 m.	21 m.	24 m.	27 m.	30 m.	33 m.	36 m.	39 m.
Minimax	14	3, -1	14, -11	9, -7	10, -5	7, -6	11, -11	9, -7	13, -13	3, -6	3, -23	8, -1	3, -8
CVaR(0.05)	14	2, 0	12, -9	9, -7	10, -5	7, -6	11, -11	9, -7	13, -13	3, -6	10, -19	8, -11	7, -5
CVaR(0.1)	18	9, -9	12, -7	5, -4	8, -6	13, -12	8, -8	15, -13	2, -7	0, -24	11, -16	12, -13	8, -9
CVaR(0.25)	11	2, -1	10, -4	10, -6	8, -8	6, -6	10, -6	8, -8	12, -9	6, -5	2, -15	9, -6	6, -6
CVaR(0.5)	17	2, -3	14, -14	17, -14	9, -4	7, -5	8, -12	8, -9	10, -4	10, -10	5, -12	6, -6	4, -5
CVaR(0.75)	17	3, -5	4, -5	4, -3	7, -2	7, -6	1, -10	8, -2	6, -2	8, -10	6, -11	7, -8	3, -4
GMD	17	1, -6	7, -5	4, -3	10, -3	5, -7	4, -8	10, -4	6, -2	8, -11	5, -9	6, -5	3, -7
WCVaR(AGD)	1	0, 0	0, 0	0, 0	0, 0	0, 0	14, 0	6, -4	7, -3	6, -6	4, -9	5, -4	3, -6
WCVaR(AGS)	1	0, 0	0, 0	0, 0	0, 0	0, 0	15, 0	9, -4	8, -6	8, -9	5, -10	6, -5	2, -6
WCVaR(AGT)	1	0, 0	0, 0	0, 0	0, 0	0, 0	16, 0	8, -5	9, -4	5, -9	5, -10	6, -2	2, -6
WCVaR(TG2)	16	7, -9	9, -9	7, -6	10, -3	7, -6	12, -9	7, -9	12, -13	1, -1	7, -13	12, -15	7, -10
WCVaR(TG3)	15	5, -2	5, -5	9, -6	9, -5	6, -5	8, -5	7, -8	7, -8	6, -5	2, -9	9, -7	4, -6

in the models behavior under such circumstances. Therefore, in order to provide a better analysis and comparison of the proposed models when the market trend is negative and thus the risk control may be relevant, we have decided to add some computational results on the period (2000-2002). During this period the Italian market has shown an impressive and continuous down-turn (see Figure 3) with the MIB30 index reaching its highest level 50467 on 10.03.2000 and its lowest level 21546 on 4.10.2002.

The following tables provide the relevant results on the analysis of the Maximum Safety Portfolios (MSPs) selected by the different models with and without diversification enforcement using the years 2000-2001 (104 weekly returns) as in-sample period and the year 2002 as out-of-sample. As for previous results, historical data are stock prices from Milan Stock Exchange. The new data set consists of 178 securities quoted with continuity from 2000 to 2002. The meaning of tables entries is identical to those described in the former sections. Due to low weekly values, the mean return  $z$  has been converted on a yearly basis.

Notice that all the models but CVaR(0.75), GMD and the Wide WCVaR models have a mean return equal to zero. Thus, as for previous experiments, the Wide WCVaR models are still among those models with larger mean returns. The use of constraints that force diversification improves, on average, the

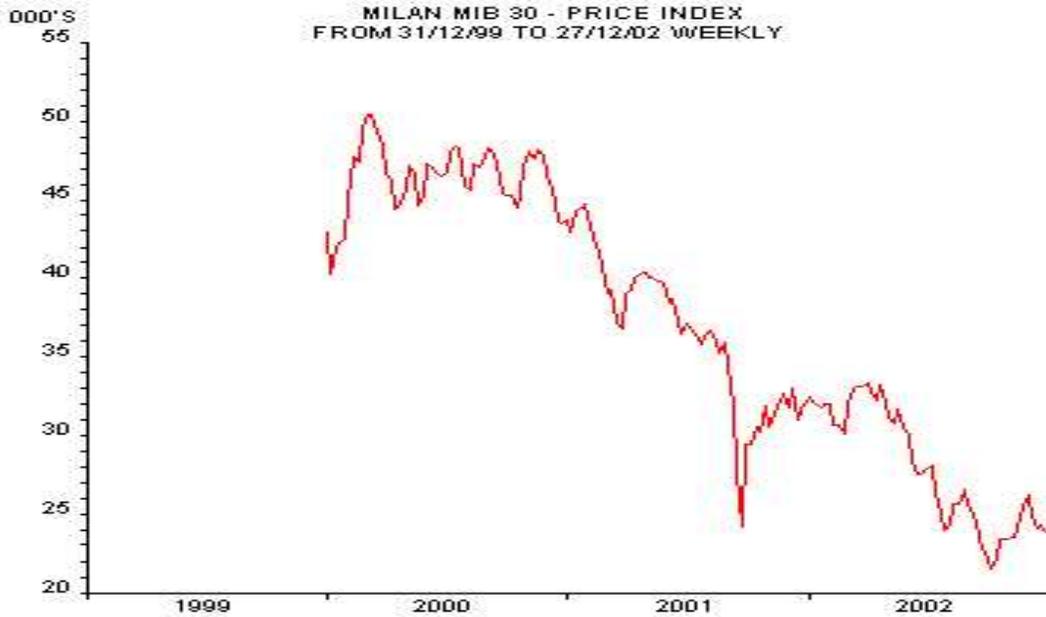


Figure 3: The Milan Stock Exchange Index (MIB30) in the years 2000–2002 (weekly quotations).

portfolios performance in terms of mean return: CVaR(0.75) shows an increase from 1.83% to 3.27%, while some models as CVaR(0.5) and CVaR(0.25) move from null to positive values. The same effect was not evident in the previous experiments.

Table 17 shows that with respect to portfolio diversification, the introduction of enforcement constraints produces an evident effect only for the Minimax model and the extremal CVaR models (CVaR(0.05) and CVaR(0.1)). As before (see Tables 4-6), the Minimax model generates some low diversified portfolios (7 securities). In contrast to previous results the basic CVaR models are not the most diversified: the CVaR(0.05) model has selected only 6 securities, while the CVaR(0.1) portfolio has 8 securities. Moreover, note that the Wide WCVaR models have selected rather diversified portfolios if we compare these results with those shown in Tables 4-6. During the period 2000-2002 no security has shown realizations with dramatically high returns, thus justifying this diversification. Finally, in only one case, i.e. for the WCVaR(TG3) model, the diversification without enforcement is larger than that obtained with additional forcing constraints. In the first part of Table 17 it is worth noticing that all the selected portfolios have a maximum share exceeding 40% (but for the model WCVaR(AGD) with 38%) and that, in two cases, namely for the models Minimax and CVaR(0.05), the maximum share is larger than 80% and 55%, respectively. For all such models we may recommend to apply the enforcement constraints (22) giving as result portfolios always containing at least 11 securities.

For the out-of-sample analysis the behavior of all the MSPs is examined in the 52 weeks following the date of each portfolio selection. The nine parameters reported in Tables 18 and 19 have the same meaning defined for previous similar tables. Again,  $r_{min}$ ,  $r_{av}$ ,  $r_{med}$  and  $r_{max}$  are expressed on a yearly basis and as per cent returns. Due to general market downward trend, all the portfolios show ex-post negative average returns: the model CVaR(0.75) has the worst performance.

By comparing Table 18 with Table 19 one may notice that, on average, the introduction of diver-

Table 17: In-sample MSPs characteristics: strong downward trend period.

models	without diversification					with diversification				
	obj. $10^{-2}$	$z$ %	div. #	shares		obj. $10^{-2}$	$z$ %	div. #	shares	
				min	max				min	max
Minimax	-0.212	0.00	7	2.06 $10^{-4}$	0.813	-0.281	0.00	14	6.01 $10^{-4}$	0.2
CVaR(0.05)	-0.189	0.00	6	2.17 $10^{-3}$	0.563	-0.231	0.00	11	8.83 $10^{-4}$	0.2
CVaR(0.1)	-0.158	0.00	8	2.38 $10^{-4}$	0.450	-0.193	0.00	16	7.62 $10^{-5}$	0.2
CVaR(0.25)	-0.107	0.00	13	1.61 $10^{-4}$	0.469	-0.129	0.11	15	2.47 $10^{-4}$	0.2
CVaR(0.5)	-0.065	0.00	13	1.69 $10^{-4}$	0.459	-0.079	0.16	13	5.48 $10^{-5}$	0.2
CVaR(0.75)	-0.031	1.83	15	1.70 $10^{-3}$	0.463	-0.033	3.27	19	1.83 $10^{-3}$	0.2
GMD	-0.050	0.10	13	2.97 $10^{-4}$	0.464	-0.058	0.60	15	1.48 $10^{-3}$	0.2
WCVaR(AGD)	-0.039	0.50	15	2.15 $10^{-4}$	0.384	-0.044	1.16	17	7.25 $10^{-5}$	0.2
WCVaR(AGS)	-0.044	0.25	15	1.46 $10^{-4}$	0.432	-0.050	0.81	16	1.24 $10^{-3}$	0.2
WCVaR(AGT)	-0.043	0.34	15	1.10 $10^{-5}$	0.495	-0.049	1.08	15	1.42 $10^{-3}$	0.2
WCVaR(TG2)	-0.131	0.00	12	7.28 $10^{-5}$	0.429	-0.158	0.00	15	1.15 $10^{-4}$	0.2
WCVaR(TG3)	-0.093	0.00	14	1.59 $10^{-4}$	0.444	-0.112	0.06	12	3.50 $10^{-3}$	0.2

Table 18: Out-of-sample results for MSPs: strong downward trend period.

Max. safety model	Without diversification enforcement								
	$r_{min}$	$r_{av}$	$r_{med}$	$r_{max}$	std	s-std	MAD	s-MAD	D-DEV
Minimax	-14.18	-0.93	-0.68	14.01	0.0009	0.0007	0.0006	0.0004	0.0029
CVaR(0.05)	-12.81	-0.73	-0.79	13.66	0.0009	0.0006	0.0007	0.0004	0.0026
CVaR(0.1)	-7.81	-0.58	-1.19	10.70	0.0008	0.0006	0.0006	0.0004	0.0016
CVaR(0.25)	-10.16	-0.88	-1.11	12.93	0.0008	0.0006	0.0006	0.0004	0.0021
CVaR(0.5)	-8.75	-0.68	-0.96	13.25	0.0008	0.0006	0.0006	0.0004	0.0018
CVaR(0.75)	-48.42	-1.49	0.67	42.82	0.0031	0.0026	0.0021	0.0012	0.0126
GMD	-9.97	-0.76	-0.61	13.21	0.0009	0.0007	0.0007	0.0004	0.0020
WCVaR(AGD)	-15.44	-1.01	-0.76	14.91	0.0011	0.0009	0.0008	0.0005	0.0032
WCVaR(AGS)	-12.77	-0.80	-0.33	12.44	0.0010	0.0008	0.0008	0.0005	0.0026
WCVaR(AGT)	-17.13	-0.94	-0.42	14.57	0.0012	0.0010	0.0009	0.0005	0.0036
WCVaR(TG2)	-7.61	-0.63	-0.59	11.56	0.0008	0.0006	0.0006	0.0004	0.0015
WCVaR(TG3)	-8.98	-0.77	-1.18	12.42	0.0008	0.0006	0.0006	0.0004	0.0018

sification enforcement may result in portfolios with larger ex-post dispersion. This is especially true for those models whose dispersion was already high without enforcement (see, for instance, the models CVaR(0.75) and CVaR(0.5) in Table 18). When we consider the models with diversification enforcement (Table 19), the model WCVaR(TG3) has the highest average return. As for previous computational results, the GMD model (which is the computationally most complex) is outperformed in terms of average returns and dispersion by the Tail WCVaR models.

Tables 20 and 21 show the ex-post cumulative portfolio returns for the case without and with the diversification enforcement, respectively. Additionally, the cumulative performances of the index MIB30 has been introduced in both tables. The weighted CVaR models show a very stable ex-post performance always outperforming GMD model. Moreover, the Wide WCVaR models also outperform extremal CVaR models when diversification enforcement is introduced (see Table 21). Actually, except for a short period of strong increase of the MIB30 index, all the models outperform the index. When comparing Table 20 to Table 21 it is evident how diversification enforcement has, on average, positively contributed to improve all the portfolios performance with the only exception of extremal CVaR portfolios whose performances are worsen with respect to the case without diversification.

Table 19: Out-of-sample results for MSPs: strong downward trend period.

Max. safety model	With diversification enforcement								
	$r_{min}$	$r_{av}$	$r_{med}$	$r_{max}$	std	s-std	MAD	s-MAD	D-DEV
Minimax	-22.28	-0.96	-0.50	20.97	0.0013	0.0010	0.0009	0.0005	0.0048
CVaR(0.05)	-12.44	-1.23	-1.27	9.63	0.0008	0.0007	0.0006	0.0004	0.0025
CVaR(0.1)	-9.78	-0.97	-1.29	9.13	0.0008	0.0006	0.0006	0.0004	0.0020
CVaR(0.25)	-9.40	-0.61	-0.46	7.70	0.0008	0.0006	0.0006	0.0004	0.0019
CVaR(0.5)	-12.04	-0.75	-0.59	7.89	0.0008	0.0007	0.0006	0.0004	0.0025
CVaR(0.75)	-62.78	-1.96	-0.07	68.23	0.0044	0.0036	0.0030	0.0017	0.0188
GMD	-13.35	-0.89	-0.32	13.77	0.0011	0.0009	0.0008	0.0005	0.0027
WCVaR(AGD)	-24.45	-1.31	-0.57	22.34	0.0016	0.0013	0.0012	0.0007	0.0054
WCVaR(AGS)	-15.93	-0.95	-0.31	16.85	0.0013	0.0010	0.0009	0.0006	0.0033
WCVaR(AGT)	-23.55	-1.22	-0.19	21.53	0.0016	0.0013	0.0011	0.0007	0.0051
WCVaR(TG2)	-10.16	-0.77	-0.55	7.96	0.0008	0.0006	0.0006	0.0004	0.0021
WCVaR(TG3)	-9.82	-0.57	-0.61	8.30	0.0007	0.0006	0.0006	0.0003	0.0020

Table 20: Cumulative out-of-sample results: strong downward trend period.

models	Without diversification enforcement												
	4 w.	8 w.	12 w.	16 w.	20 w.	24 w.	28 w.	32 w.	36 m.	40 w.	44 w.	48 w.	52 w.
Minimax	3.35	-3.41	4.23	1.55	0.74	-3.25	-7.25	-7.97	-7.81	-10.04	-5.94	-2.94	-0.93
CVaR(0.05)	2.51	-2.99	5.89	2.72	2.21	-0.70	-4.58	-6.96	-6.81	-9.18	-5.22	-2.12	-0.73
CVaR(0.1)	0.43	-4.33	5.44	2.86	2.63	-0.18	-4.44	-6.40	-6.42	-8.71	-5.17	-1.88	-0.58
CVaR(0.25)	-2.68	-8.19	-3.22	-3.62	-4.73	-5.08	-8.33	-9.31	-9.43	-13.40	-8.57	-3.29	-0.88
CVaR(0.5)	-2.13	-4.90	0.22	0.22	-1.60	-2.78	-5.85	-7.69	-7.74	-12.29	-7.66	-2.79	-0.68
CVaR(0.75)	17.33	15.44	45.49	41.16	59.95	38.90	24.21	10.89	4.31	-15.59	-5.71	-1.24	-1.51
GMD	-0.66	-5.09	2.02	0.76	0.76	-0.26	-4.38	-6.71	-7.36	-11.91	-7.14	-2.48	-0.76
WCVaR(AGD)	1.27	-4.44	4.99	4.13	6.31	2.31	-2.74	-5.48	-6.84	-14.40	-8.10	-3.13	-1.01
WCVaR(AGS)	1.08	-3.04	5.50	4.14	4.98	2.20	-2.73	-5.46	-6.43	-12.11	-6.90	-2.40	-0.80
WCVaR(AGT)	0.47	-3.79	4.66	3.65	5.22	3.65	-2.24	-5.68	-7.03	-13.90	-7.90	-2.69	-0.94
WCVaR(TG2)	-0.36	-5.68	1.84	-0.16	-0.22	-1.59	-4.64	-6.43	-6.91	-9.72	-5.92	-2.09	-0.63
WCVaR(TG3)	-2.21	-7.20	-1.09	-2.04	-2.94	-3.48	-6.82	-8.46	-8.57	-12.29	-7.67	-2.84	-0.78
MIB30	-26.71	-95.69	174.00	181.75	-45.95	-98.92	-99.43	-99.17	-98.68	-99.47	-94.31	-64.90	-26.45

The additional experimental analysis over the period 2000-2002 has allowed us to draw the following main conclusions. First, during strongly negative market trend the weighted CVaR models have, on average, performed better than the GMD, the Minimax and the extremal CVaR models. Actually, in terms of cumulative returns all the models have beaten the MIB30 index performance. Second, during strongly negative market trend the Wide WCVaR models show a more stable behavior than in positive trend period. In the latter case they may need diversification enforcement. Generally, the diversification enforcement turns out to be necessary and effective for all the models rather during unstable market trends (typically characterized by quick changes of market directions, as for previous experiments) than during strong downward periods where for some models, as for the extremal CVaR models, diversification enforcement has made performances poorer.

Table 21: Cumulative out-of-sample results: strong downward trend period.

models	With diversification enforcement												
	4 w.	8 w.	12 w.	16 w.	20 w.	24 w.	28 w.	32 w.	36 m.	40 w.	44 w.	48 w.	52 w.
Minimax	-0.75	-4.82	12.72	11.80	10.55	1.85	-4.21	-7.25	-6.69	-11.67	-5.87	-2.31	-0.96
CVaR(0.05)	-4.84	-9.82	-1.42	-0.97	-4.50	-8.01	-11.31	-12.39	-11.18	-16.41	-11.07	-5.25	-1.23
CVaR(0.1)	-5.40	-10.72	0.10	0.42	-2.35	-8.70	-10.86	-11.93	-10.62	-15.59	-9.66	-4.33	-0.97
CVaR(0.25)	-1.62	-6.07	2.01	4.28	1.91	-3.73	-6.71	-7.62	-7.01	-12.35	-6.88	-2.79	-0.61
CVaR(0.5)	-2.31	-5.92	0.14	2.34	-0.97	-3.33	-6.26	-7.56	-7.33	-13.61	-8.25	-3.54	-0.75
CVaR(0.75)	29.69	22.69	82.34	78.72	117.34	63.31	43.86	23.26	11.48	-16.54	-4.62	-0.52	-2.01
GMD	0.80	-2.78	6.71	7.48	7.35	2.54	-1.51	-4.45	-5.77	-13.97	-7.52	-3.08	-0.89
WCVaR(AGD)	4.66	0.53	12.61	12.57	15.39	7.67	2.73	-2.26	-4.96	-16.82	-8.60	-3.70	-1.31
WCVaR(AGS)	2.80	-0.74	11.03	11.14	11.93	5.67	0.88	-2.62	-4.92	-14.34	-7.47	-2.93	-0.95
WCVaR(AGT)	4.61	0.71	13.05	13.34	17.08	9.66	3.25	-2.18	-4.68	-16.98	-8.45	-3.41	-1.23
WCVaR(TG2)	-3.09	-7.69	1.66	3.54	-0.08	-5.82	-8.53	-9.56	-8.50	-14.06	-8.42	-3.61	-0.78
WCVaR(TG3)	-1.93	-5.78	1.83	3.83	0.72	-1.63	-4.97	-6.59	-6.44	-11.68	-6.82	-2.69	-0.57
MIB30	-26.71	-95.69	174.00	181.75	-45.95	-98.92	-99.43	-99.17	-98.68	-99.47	-94.31	-64.90	-26.45

## 5 Concluding remarks

In this paper we have studied LP solvable portfolio optimization models based on extensions of the Conditional Value at Risk (CVaR) measure. The models use multiple CVaR measures thus allowing for more detailed risk aversion modeling. All the studied models are SSD consistent and may be considered some approximations to the Gini’s mean difference with the advantage of being computationally much simpler than the GMD model itself. Our analysis has been focused on the weighted CVaR measures defined as simple combinations of a very few CVaR measures. We have introduced two specific types of weights settings which relate the WCVaR measure to the Gini’s mean difference (the Wide WCVaR) and its tail version (the Tail WCVaR). This allows us to use a few tolerance levels as only parameters specifying the entire WCVaR measures while the corresponding weights are automatically predefined by the requirements of the corresponding Gini’s measures.

Our experimental analysis of the models performance on the real-life data from the Milan Stock Exchange has confirmed their attractiveness. The weighted CVaR models have usually performed better than the GMD itself, the Minimax or the extremal CVaR models. These promising results show a need for further comprehensive experimental studies analyzing practical performances of the weighted CVaR models within specific areas of financial applications. It is important to notice that although the quantile risk measures (VaR and CVaR) were introduced in banking as extreme risk measures for very small tolerance levels (like  $\beta = 0.05$ ), for the portfolio optimization good results have been provided by rather larger tolerance levels. Additional experimental analysis over the period 2000-2002 with strongly negative market trend has confirmed good achievements of the weighted CVaR models. They have, on average, performed better than the GMD, the Minimax and the extremal CVaR models. In terms of cumulative returns all the models have outperformed the MIB30 index.

While the Tail WCVaR models have always generated well diversified portfolios, the Wide WCVaR models require some diversification enforcement to avoid too small portfolios. Our experiments have also confirmed effectiveness of our CVaR based technique for a direct diversification enforcement. Although, the diversification enforcement turns out to be necessary and effective rather during unstable market trends (typically characterized by quick changes of market directions, as for previous experiments) than

during strong downward periods.

## Appendix

The spectral risk measures have been shown to be coherent in the sense of Artzner et al. (1999). These coherence axioms are based on the standard ‘monotonicity’ ( $X \geq Y$  a.s.  $\Rightarrow \rho(X) \leq \rho(Y)$ ). Since the measures represent the dual theory of choice (Yaari (1987)), they should also satisfy stronger monotonicity related to the SSD dominance. Indeed, the following SSD consistency results can be shown.

**Theorem 3** *If a nonnegative real function  $\phi$  on the interval  $[0, 1]$  is weakly decreasing (i.e.,  $0 \leq \alpha_1 < \alpha_2 \leq 1$  implies  $\phi(\alpha_1) \geq \phi(\alpha_2)$ ), then the spectral measure  $M_\phi(\mathbf{x}) = \int_0^1 \phi(\alpha) F_{\mathbf{x}}^{(-1)}(\alpha) d\alpha$  is SSD consistent, i.e.,*

$$R_{\mathbf{x}'} \succeq_{SSD} R_{\mathbf{x}''} \quad \Rightarrow \quad M_\phi(\mathbf{x}') \geq M_\phi(\mathbf{x}'').$$

**Proof.** Note that

$$M_\phi(\mathbf{x}') - M_\phi(\mathbf{x}'') = \int_0^1 \phi(\alpha) (F_{\mathbf{x}'}^{(-1)}(\alpha) - F_{\mathbf{x}''}^{(-1)}(\alpha)) d\alpha$$

and integrating by parts one gets

$$M_\phi(\mathbf{x}') - M_\phi(\mathbf{x}'') = \phi(1)(F_{\mathbf{x}'}^{(-2)}(1) - F_{\mathbf{x}''}^{(-2)}(1)) - \int_0^1 (F_{\mathbf{x}'}^{(-2)}(\alpha) - F_{\mathbf{x}''}^{(-2)}(\alpha)) d\phi(\alpha) \quad (23)$$

If  $R_{\mathbf{x}'} \succeq_{SSD} R_{\mathbf{x}''}$ , then  $F_{\mathbf{x}'}^{(-2)}(\alpha) \geq F_{\mathbf{x}''}^{(-2)}(\alpha)$  for all  $\alpha \in [0, 1]$  and, in particular,  $F_{\mathbf{x}'}^{(-2)}(1) - F_{\mathbf{x}''}^{(-2)}(1) = \mu(\mathbf{x}') - \mu(\mathbf{x}'') \geq 0$ . Hence, due to assumed properties of function  $\phi$ , one gets  $M_\phi(\mathbf{x}') - M_\phi(\mathbf{x}'') \geq 0$ , which completes the proof.  $\square$

**Theorem 4** *If a nonnegative real function  $\phi$  on the interval  $[0, 1]$  is strictly decreasing (i.e.,  $0 \leq \alpha_1 < \alpha_2 \leq 1$  implies  $\phi(\alpha_1) > \phi(\alpha_2)$ ), then the spectral measure  $M_\phi(\mathbf{x}) = \int_0^1 \phi(\alpha) F_{\mathbf{x}}^{(-1)}(\alpha) d\alpha$  is strictly SSD consistent, i.e.,*

$$R_{\mathbf{x}'} \succ_{SSD} R_{\mathbf{x}''} \quad \Rightarrow \quad M_\phi(\mathbf{x}') > M_\phi(\mathbf{x}'').$$

**Proof.** If  $R_{\mathbf{x}'} \succ_{SSD} R_{\mathbf{x}''}$ , then  $F_{\mathbf{x}'}^{(-2)}(\alpha) \geq F_{\mathbf{x}''}^{(-2)}(\alpha)$  for all  $\alpha \in [0, 1]$  with at least one strict inequality for some  $\alpha_0 \in (0, 1]$ . Moreover,  $F_{\mathbf{x}}^{(-2)}$  are continuous functions (Ogryczak and Ruszczyński (2002a)). Therefore, there exist  $\varepsilon > 0$  such that  $F_{\mathbf{x}'}^{(-2)}(\alpha) > F_{\mathbf{x}''}^{(-2)}(\alpha)$  for all  $\alpha \in (\alpha_0 - \varepsilon, \alpha_0]$ . Hence, due to assumed properties of function  $\phi$ , using (23) one gets  $M_\phi(\mathbf{x}') - M_\phi(\mathbf{x}'') > 0$ , which completes the proof.  $\square$

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