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Reference point method with importance weighted ordered partial achievements

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Abstract The Reference Point Method (RPM) is a very convenient technique for interactive analysis of the multiple criteria optimization problems. The interactive analysis is navigated with the commonly accepted control parameters expressing reference levels for the individual objective functions. The partial achievement functions quantify the DM satisfaction from the individual outcomes with respect to the given reference levels, while the final scalarizing achievement function is built as the augmented max–min aggregation of the partial achievements. In order to avoid inconsistencies caused by the regularization, the max–min solution may be regularized by the Ordered Weighted Averages (OWA) with monotonic weights which combines all the partial achievements allocating the largest weight to the worst achievement, the second largest weight to the second worst achievement, and so on. Further, following the concept of the Weighted OWA (WOWA), the importance weighting of several achievements may be incorporated into the RPM. Such a WOWA RPM approach uses importance weights to affect achievement importance by rescaling accordingly its measure within the distribution of achievements rather than by straightforward rescaling of achievement values. The recent progress in optimization methods for ordered averages allows one to implement the WOWA RPM quite effectively as extension of the original constraints and criteria with simple linear inequalities. There is shown that the OWA and WOWA RPM models meet the crucial requirements with respect to the efficiency of generated solutions as well as the controllability of interactive analysis by the reference levels.

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1 Introduction

Consider a decision problem defined as an optimization problem with m criteria (objective functions). In this paper, without loss of generality, it is assumed that all the criteria are maximized (that is, for each outcome, "more is better"). Hence, we consider the following multiple criteria optimization problem:

$$\max\{(f_1(\mathbf{x}), \dots, f_m(\mathbf{x})) : \mathbf{x} \in Q\}, \tag{1}$$

where $\mathbf{x}$ denotes a vector of decision variables to be selected within the feasible set $Q \subset R^n$, and $\mathbf{f}(x) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \dots, f_m(\mathbf{x}))$ is a vector function that maps the feasible set Q into the criterion space R^m. Note that neither any specific form of the feasible set Q is assumed nor any special form of criteria $f_i(\mathbf{x})$ is required. We refer to the elements of the criterion space as outcome vectors. An outcome vector $\mathbf{y}$ is attainable if it expresses outcomes of a feasible solution, i.e., $\mathbf{y} = \mathbf{f}(\mathbf{x})$ for some $\mathbf{x} \in Q$.

Model (1) only specifies that we are interested in maximization of all objective functions f_i for $i \in I = \{1, 2, \dots, m\}$. Thus it allows one only to identify (to eliminate) obviously inefficient solutions leading to dominated outcome vectors while still leaving the entire efficient set to look for a satisfactory compromise solution. In order to make the multiple criteria model operational for the decision support process, one needs to assume some solution concept well adjusted to the DM preferences. This can be achieved with the so-called quasi-satisficing approach to multiple criteria decision problems. The best formalization of the quasi-satisficing approach to multiple criteria optimization was proposed and developed mainly by Wierzbicki (1982) as the Reference Point Method (RPM). The reference point method was later extended to permit additional information from the DM and, eventually, led to efficient implementations of the so-called Aspiration/Reservation-Based Decision Support (ARBDS) approach with many successful applications (Granat and Makowski 2000; Lewandowski and Wierzbicki 1989; Ogryczak and Lahoda 1992; Wierzbicki et al. 2000).

The RPM is an interactive technique. The basic concept of the interactive scheme is as follows. The DM specifies requirements in terms of reference levels, i.e., by introducing reference (target) values for several individual outcomes. Depending on the specified reference levels, a special scalarizing achievement function is built, which may be directly interpreted as expressing utility to be maximized. Maximization of the scalarizing achievement function generates an efficient solution to the multiple criteria problem. The computed efficient solution is presented to the DM as the current solution in a form that allows comparison with the previous ones and modification of the reference levels if necessary.

The scalarizing achievement function can be viewed as a two-stage transformation of the original outcomes. First, the strictly monotonic partial achievement functions are built to measure individual performance with respect to given reference levels. Having all the outcomes transformed into a uniform scale of individual achievements, they are aggregated at the second stage to form a unique scalarization. The RPM is based on the so-called augmented (or regularized) max–min aggregation. Thus, the worst individual achievement is essentially maximized, but the optimization process is additionally regularized with the term representing the average achievement. The max–min aggregation guarantees fair treatment of all individual achievements by implementing an approximation to the Rawlsian principle of justice.

The max–min aggregation is crucial for allowing the RPM to generate all efficient solutions even for nonconvex (and particularly discrete) problems. On the other hand, the regularization is necessary to guarantee that only efficient solutions are generated. The regularization by the average achievement is easily implementable, but it may disturb the basic max–min model. Actually, the only consequent regularization of the max–min aggregation is the lex-min order, but the lexicographic approaches result in complicated RPM models (Ogryczak 2008, 2009b), and they do not introduce any explicit scalarizing achievement function which could be directly interpreted as expressing utility to be maximized. In order to get such an analytical form, one needs to replace the lexicographic (preemptive) optimization of the ordered achievements with the so-called Ordered Weighted Averaging (OWA) aggregation (Yager 1988). The latter combines all the partial achievements allocating the largest weight to the worst achievement, the second largest weight to the second worst achievement, the third largest weight to the third worst achievement, and so on. It turns out that a direct use of the OWA aggregation to the partial achievements improves the RPM model without any necessity of employment of the lexicographic techniques. Moreover, the recent progress in optimization methods for ordered averages (Ogryczak and Śliwiński 2003) allows one to implement the OWA RPM quite effectively. We show that such RPM model meets the crucial requirements with respect to the efficiency of the generated solution as well as the controllability of interaction analysis by the reference levels as the only control parameters.

The standard RPM models allow weighting of several achievements only by straightforward rescaling of the achievement values (Ruiz et al. 2009). However, there are many decision situations where there is a need for importance weighting of achievements. This can be related to the involvement of multiple DMs to specify the preferences like in procurement (Teich et al. 2006) or in more complex policy decisions involving many stakeholders (Makowski et al. 2009). The OWA RPM model enables us to introduce importance weights to affect an achievement importance by rescaling accordingly its measure within the distribution of all achievements. This concept (Ogryczak and Kozłowski 2008) can be formalized with the so-called Weighted OWA (WOWA) aggregations (Torra 1997; Torra and Narukawa 2007) of the partial achievements. The paper analyzes both the theoretical and implementation issues of the WOWA enhanced RPM. We show that the WOWA RPM model meets the crucial requirements with respect to the efficiency of the generated solution as well as the controllability of interaction analysis by the reference levels.

The paper is organized as follows. In the next section the scalarizing achievement functions are discussed, and there is introduced and analyzed the OWA refinement of the RPM. The OWA RPM model is further extended in Sect. 3 to accommodate the importance weights following the WOWA methodology. Linear Programming computational model for the WOWA RPM method is introduced, and the crucial properties of the method are shown. In Sect. 4 an illustrative example is presented.

2 Scalarizations of the RPM and ordered partial achievements

While building the scalarizing achievement function, the following properties of the preference model are assumed. First of all, for any individual outcome y_i, more is preferred to less (maximization). To meet this requirement, the function must be strictly increasing with respect to each outcome. Second, a solution with all individual outcomes y_i satisfying the corresponding reference levels is preferred to any solution with at least one individual outcome worse (smaller) than its reference level. That means, the scalarizing achievement function maximization must enforce reaching the reference levels prior to further improving of criteria. Thus, similar to the goal programming approaches, the reference levels are treated as the targets, but following the quasi-satisficing approach, they are interpreted consistently with basic concepts of efficiency in the sense that the optimization is continued even when the target point has been already reached.

The generic scalarizing achievement function takes the following form (Wierzbicki 1982):

$$S(\mathbf{y}) = \min_{1 \le i \le m} \{s_i(y_i)\} + \frac{\varepsilon}{m} \sum_{i=1}^{m} s_i(y_i), \tag{2}$$

where ε is an arbitrary small positive number, and $s_i : R \to R$, $i = 1, 2, \ldots, m$, are the partial achievement functions measuring actual achievement of the individual outcomes y_i with respect to the corresponding reference levels. Let a_i denote the partial achievement for the ith outcome ($a_i = s_i(y_i)$), and $\mathbf{a} = (a_1, a_2, \ldots, a_m)$ represent the achievement vector. The scalarizing achievement function (2) is, essentially, defined by the worst partial (individual) achievement but additionally regularized with the sum of all partial achievements. The regularization term is introduced only to guarantee the solution efficiency in the case where the maximization of the main term (the worst partial achievement) results in a nonunique optimal solution. Due to combining two terms with arbitrarily small parameter ε, formula (2) is easily implementable and provides a direct interpretation of the scalarizing achievement function as expressing utility.

Various functions s_i provide a wide modeling environment for measuring partial achievements (Wierzbicki 1986; Wierzbicki et al. 2000; Miettinen and Mäkelä 2002). The basic RPM model is based on a single vector of the reference levels, the aspiration vector $\mathbf{r}^a$, and the piecewise linear functions s_i. It can be effectively applied to multiple criteria linear programming problems (Steuer 1986), nonlinear (Miettinen 1999) or discrete.

Real-life applications of the RPM methodology usually deal with more complex partial achievement functions defined with more than one reference point (Wierzbicki et al. 2000) which enriches the preference models and simplifies the interactive analysis. In particular, the models taking advantages of two reference vectors, vector of aspiration levels $\mathbf{r}^a$ and vector of reservation levels $\mathbf{r}^r$ (Lewandowski and Wierzbicki 1989), are used, thus allowing the DM to specify requirements by introducing acceptable and required values for several outcomes. The partial achievement function s_i can be interpreted then as a measure of the DM's satisfaction with the current value of outcome the ith criterion. It is a strictly increasing function of outcome y_i with value $a_i = 1$ if $y_i = r_i^a$, and $a_i = 0$ for $y_i = r_i^r$. Thus the partial achievement functions map the outcomes values onto a normalized scale of the DM's satisfaction. Various functions can be built meeting those requirements. We use the piecewise linear partial achievement function introduced in an implementation of the ARBDS system for the multiple criteria transshipment problems with facility location (Ogryczak et al. 1992):

$$s_i(y_i) = \begin{cases} \gamma(y_i - r_i^r)/(r_i^a - r_i^r), & y_i \le r_i^r, \\ (y_i - r_i^r)/(r_i^a - r_i^r), & r_i^r < y_i < r_i^a, \\ \alpha(y_i - r_i^a)/(r_i^a - r_i^r) + 1, & y_i \ge r_i^a, \end{cases} \tag{3}$$

where α and γ are arbitrarily defined parameters satisfying $0 < \alpha < 1 < \gamma$. Parameter α represents additional increase of the DM's satisfaction over level 1 when a criterion generates outcomes better than the corresponding aspiration level. On the other hand, parameter $\gamma > 1$ represents dissatisfaction connected with outcomes worse than the reservation level.

For outcomes between the reservation and the aspiration levels, the partial achievement function s_i can be interpreted as a membership function μ_i for a fuzzy target. However, such a membership function remains constant with value 1 for all outcomes greater than the corresponding aspiration level and with value 0 for all outcomes below the reservation level. Hence, the fuzzy membership function is neither strictly monotonic nor concave and thus not representing typical utility for a maximized outcome. The partial achievement function (3) can be viewed as an extension of the fuzzy membership function to a strictly monotonic and concave utility. Since partial achievement function (3) is strictly increasing and concave, it can be expressed in the form

$$s_i(y_i) = \min\left\{\gamma \frac{y_i - r_i^r}{r_i^a - r_i^r}, \frac{y_i - r_i^r}{r_i^a - r_i^r}, \alpha \frac{y_i - r_i^a}{r_i^a - r_i^r} + 1\right\} \tag{4}$$

which guarantees LP computability with respect to outcomes y_i. Finally, maximization of the entire scalarizing achievement function (2) can be implemented by the following auxiliary LP constraints:

$$\max \quad z + \frac{\varepsilon}{m} \sum_{i=1}^{m} a_i$$

$$
\begin{aligned}
\text{s.t.} \quad & z \le a_i, && \forall i \in I, \\
& a_i \le \gamma\big(y_i - r_i^r\big)/\big(r_i^a - r_i^r\big), && \forall i \in I, \\
& a_i \le \big(y_i - r_i^r\big)/\big(r_i^a - r_i^r\big), && \forall i \in I, \\
& a_i \le \alpha\big(y_i - r_i^a\big)/\big(r_i^a - r_i^r\big) + 1, && \forall i \in I, \\
& \mathbf{x} \in Q, \quad y_i = f_i(\mathbf{x}), && \forall i \in I,
\end{aligned}
\tag{5}
$$

where a_i for $i = 1, \ldots, m$ and z are unbounded variables introduced to represent values of several partial achievement functions and their minimum, respectively.

For any reference levels $r_i^a > r_i^r$ (attainable or not), the RPM always generates an efficient solution. That means, if $\bar{\mathbf{x}} \in Q$, together with $\bar{\mathbf{a}}$, $\bar{\mathbf{y}}$, $\bar{z}$, is an optimal solution of problem (5), then $\bar{\mathbf{x}}$ is an efficient solution of the corresponding multiple criteria optimization problem (1). Moreover, the RPM supports controllability of the interactive process by the reference levels in the sense that various efficient solutions can be selected by appropriate setting of the reference levels. Actually, only lexicographic forms of the RPM (Ogryczak 1994, 2001; Ogryczak and Lahoda 1992) maintain the complete controllability guaranteeing that for any efficient solution, there exist reference levels allowing one to identify that solution as optimal. While using the analytic forms of the scalarizing achievement function, the regularization term causes that only properly efficient solutions (Geoffrion 1968) with bounded tradeoffs can be generated (Kaliszewski 1994; Wierzbicki 1999). Recall that an efficient solution $\bar{\mathbf{x}} \in Q$ with corresponding outcome vector $\bar{\mathbf{y}} = \mathbf{f}(\bar{\mathbf{x}})$ is properly efficient with tradeoffs bounded by Δ if and only if, for any attainable outcome vector $\mathbf{y}$, the implication

$$
y_i > \bar{y}_i \text{ and } y_k < \bar{y}_k \quad \Rightarrow \quad \frac{y_i - \bar{y}_i}{\bar{y}_k - y_k} \le \Delta \tag{6}
$$

is valid for any $i, k \in I$. Each properly efficient solution $\bar{\mathbf{x}}$ with tradeoffs bounded by $\Delta = \frac{m+\varepsilon}{\alpha(m-1)\varepsilon}$ is an optimal solution to the RPM problem (5) for aspiration levels $r_i^a = f_i(\bar{\mathbf{x}})$ and reservation levels $r_i^r = f_i(\bar{\mathbf{x}}) - 1$, $i = 1, 2, \ldots, m$.

The crucial properties of the RPM are related to the max–min aggregation of partial achievements, while the regularization is only introduced to guarantee the aggregation monotonicity. Unfortunately, the distribution of achievements may make the max–min criterion partially passive when one specific achievement is relatively very small for all the solutions. Maximization of the worst achievement may then leave all other achievements unoptimized. Nevertheless, the selection is then made according to linear aggregation of the regularization term instead of the max–min aggregation, thus destroying the preference model of the RPM. This can be illustrated with an example of a simple discrete problem of seven alternative feasible solutions to be selected according to six criteria. Table 1 presents six partial achievements for all the solutions where the partial achievements have been defined according to the aspiration/reservation model (3), thus allocating 1 to outcomes reaching the corresponding aspiration level. All the solutions are efficient. Solution S1 to S5 oversteps the aspiration levels (achievement values 1.2) for four of the first five criteria while failing to reach one of them and the aspiration level for the sixth criterion as well (achievement values 0.3). Solution S6 meets the aspiration levels (achievement values 1.0)

Table 1 Sample achievements with passive max–min criterion

Sol.	a_1	a_2	a_3	a_4	a_5	a_6	min	$\sum$
S1	0.3	1.2	1.2	1.2	1.2	0.3	0.3	5.4
S2	1.2	0.3	1.2	1.2	1.2	0.3	0.3	5.4
S3	1.2	1.2	0.3	1.2	1.2	0.3	0.3	5.4
S4	1.2	1.2	1.2	0.3	1.2	0.3	0.3	5.4
S5	1.2	1.2	1.2	1.2	0.3	0.3	0.3	5.4
S6	1.0	1.0	1.0	1.0	1.0	0.3	0.3	5.3
S7	0.3	0.3	0.3	1.0	0.6	1.0	0.3	3.5

for the first five criteria while failing to reach only the aspiration level for the sixth criterion (achievement values 0.3). All the solutions generate the same worst achievement value 0.3, and the final selection of the RPM depends on the total achievement (regularization term). Actually, one of solutions S1 to S5 will be selected as better than S6.

In order to avoid inconsistencies caused by the regularization, the max–min solution may be regularized lexicographically (Ogryczak 2008) or according to the ordered averaging rules (Yager 1988). This is mathematically formalized as follows. Within the space of achievement vectors, we introduce the map $\Theta = (\theta_1, \theta_2, \dots, \theta_m)$ which orders the coordinates of achievements vectors in a nonincreasing order, i.e., $\Theta(a_1, a_2, \dots, a_m) = (\theta_1(\mathbf{a}), \theta_2(\mathbf{a}), \dots, \theta_m(\mathbf{a}))$ iff there exists a permutation τ such that $\theta_i(\mathbf{a}) = a_{\tau(i)}$ for all i and $\theta_1(\mathbf{a}) \ge \theta_2(\mathbf{a}) \ge \cdots \ge \theta_m(\mathbf{a})$. The standard max–min aggregation depends on maximization of $\theta_m(\mathbf{a})$ and ignores the values of $\theta_i(\mathbf{a})$ for $i \le m-1$. In order to take into account all the achievement values, one needs to maximize the weighted combination of the ordered achievements, thus representing the so-called Ordered Weighted Averaging (OWA) aggregation (Yager 1988). Note that the weights are then assigned to the specific positions within the ordered achievements rather than to the partial achievements themselves. With the OWA aggregation, one gets the following RPM model:

$$\max\left\{\sum_{i=1}^{m} w_i\theta_i(\mathbf{a}) : \mathbf{x} \in Q, a_i = s_i\bigl(f_i(\mathbf{x})\bigr)\ \forall i \in I\right\}, \tag{7}$$

where $w_1 < w_2 < \cdots < w_m$ are positive and strictly increasing weights. Note that the standard RPM model with the scalarizing achievement function (2) can be expressed as the OWA model $\max((1 + \frac{\varepsilon}{m})\theta_m(\mathbf{a}) + \frac{\varepsilon}{m}\sum_{i=1}^{m-1}\theta_i(\mathbf{a}))$. Hence, the standard RPM model exactly represents the OWA aggregation (7) with strictly increasing weights in the case of $m = 2$ ($w_1 = \varepsilon/2 < w_2 = 1 + \varepsilon/2$). For $m > 2$, it abandons the differences in weighting the largest achievement, the second largest one, etc. ($w_1 = \cdots = w_{m-1} = \varepsilon/m$). The OWA RPM model (7) allows one to distinguish all the weights by introducing increasing series. One may notice in Table 2 that application of increasing weights $\mathbf{w} = (0.02, 0.03, 0.05, 0.15, 0.25, 0.5)$ within the OWA RPM enables selection of solution S6 from Table 1. Actually, the OWA weights should be significantly increasing to represent regularization of the max–min order. When differences among weights tend to infinity, the OWA aggregation approximates the leximin ranking of

Table 2 Ordered achievements values

Sol.	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	$A_{\mathbf{w}}$
S1	1.2	1.2	1.2	1.2	0.3	0.3	0.525
S2	1.2	1.2	1.2	1.2	0.3	0.3	0.525
S3	1.2	1.2	1.2	1.2	0.3	0.3	0.525
S4	1.2	1.2	1.2	1.2	0.3	0.3	0.525
S5	1.2	1.2	1.2	1.2	0.3	0.3	0.525
S6	1.0	1.0	1.0	1.0	1.0	0.3	0.650
S7	1.0	1.0	0.6	0.3	0.3	0.3	0.305
w	0.02	0.03	0.05	0.15	0.25	0.5	

the ordered outcome vectors (Yager 1997), thus allowing the corresponding OWA RPM to approximate the leximin (nucleolar) RPM concept (Kostreva et al. 2004; Ogryczak 2008). However, any finite differences small enough to allow for numerical computation of the OWA values provide the analytic form of the scalarizing achievement function. Hence, we recommend the use of geometric increasing series to define the OWA RPM weights w_i. Note that, similar to the regularization parameter ε in the standard scalarizing achievement function (2), the OWA weights w_i are the RPM internal parameters rather than the control parameters to be set by the DM during the interactive process.

An important advantage of the RPM depends on its easy implementation as an expansion of the original multiple criteria model. Actually, even complicated partial achievement functions of the form (3) are strictly increasing and concave, thus allowing for implementation of the entire RPM model (2) by an LP expansion (Ogryczak et al. 1992).

The OWA aggregation is obviously a piecewise linear function since it remains linear within every area of the fixed order of arguments. The ordered achievements used in the OWA aggregation are, in general, hard to implement due to the pointwise ordering. However, its optimization can be implemented with the use of the cumulated ordered achievements $\bar{\theta}_k(\mathbf{a}) = \sum_{i=1}^{k} \theta_{m-i+1}(\mathbf{a})$ expressing, respectively: the worst (smallest) achievement, the total of the two worst achievements, the total of the three worst achievements, etc. Indeed,

$$\sum_{i=1}^{m} w_i \theta_i(\mathbf{a}) = \sum_{i=1}^{m} w_i' \bar{\theta}_i(\mathbf{a}),$$

where $w_i' = w_{m-i+1} - w_{m-i}$ for $i = 1, \ldots, m-1$, and $w_m' = w_1$. This simplifies dramatically the optimization problem since quantities $\bar{\theta}_k(\mathbf{a})$ can be optimized without use of any integer variables (Ogryczak and Śliwiński 2003). First, let us notice that for any given vector $\mathbf{a}$, the cumulated ordered value $\bar{\theta}_k(\mathbf{a})$ can be found as the optimal value of the following LP problem:

$$\bar{\theta}_k(\mathbf{a}) = \min_{u_{ik}} \left\{ \sum_{i=1}^{m} a_i u_{ik} : \sum_{i=1}^{m} u_{ik} = k,\ 0 \le u_{ik} \le 1\ \forall i \right\}. \tag{8}$$

The above problem is an LP for a given outcome vector $\mathbf{a}$, while it becomes nonlinear for $\mathbf{a}$ being a vector of variables. This difficulty can be overcome by taking advantage of the LP dual to (8). Introducing a dual variable t_k corresponding to the equation $\sum_{i=1}^m u_{ik} = k$ and variables d_{ik} corresponding to upper bounds on u_{ik}, one gets the following LP dual of problem (8):

$$\bar{\theta}_k(\mathbf{a}) = \max_{t_k, d_{ik}} \left\{ kt_k - \sum_{i=1}^m d_{ik} : a_i \geq t_k - d_{ik}, d_{ik} \geq 0 \ \forall i \right\}. \tag{9}$$

Due the duality theory, for any given vector $\mathbf{a}$, the cumulated ordered coefficient $\bar{\theta}_k(\mathbf{a})$ can be found as the optimal value of the above LP problem.

Taking advantages of the LP expression (9) for $\bar{\theta}_i$, the entire OWA aggregation of the partial achievement functions (7) can be expressed in terms of LP. Moreover, in the case of concave piecewise linear partial achievement functions, as typically used in the RPM approaches, the resulting formulation extends the original constraints and criteria with auxiliary linear inequalities. In particular, for strictly increasing and concave partial achievement functions (3), the OWA RPM can be expressed in the form

$$\begin{aligned}
\max \quad & \sum_{k=1}^m w'_k z_k \\
\text{s.t.} \quad & z_k = kt_k - \sum_{i=1}^m d_{ik}, && \forall k \in I, \\
& \mathbf{x} \in Q, \qquad y_i = f_i(\mathbf{x}), && \forall i \in I, \\
& a_i \geq t_k - d_{ik}, \qquad d_{ik} \geq 0, && \forall i, k \in I \\
& a_i \leq \gamma\left(y_i - r_i^r\right)/\left(r_i^a - r_i^r\right), && \forall i \in I, \\
& a_i \leq \left(y_i - r_i^r\right)/\left(r_i^a - r_i^r\right), && \forall i \in I, \\
& a_i \leq \alpha\left(y_i - r_i^a\right)/\left(r_i^a - r_i^r\right) + 1, && \forall i \in I.
\end{aligned} \tag{10}$$

Theorem 1 *For any reference levels $r_i^a > r_i^r$, $i \in I$, and any positive weights $\mathbf{w}'$, if $\bar{\mathbf{x}} \in Q$, together with $\bar{\mathbf{a}}, \bar{\mathbf{y}}, \bar{\mathbf{z}}, \bar{\mathbf{t}}, \bar{\mathbf{d}}$, is an optimal solution of the OWA RPM problem* (10), *then $\bar{\mathbf{x}}$ is an efficient solution of the corresponding multiple criteria optimization problem* (1).

Proof Let $(\bar{\mathbf{x}}, \bar{\mathbf{a}}, \bar{\mathbf{y}}, \bar{\mathbf{z}}, \bar{\mathbf{t}}, \bar{\mathbf{d}})$ be an optimal solution of problem (10) with some positive weighting vector $\mathbf{w}'$. Following formulas (4) and (9), one may notice that problem (10) is equivalent to the OWA RPM problem (7) with partial achievement functions s_i defined by (3) and positive strictly increasing OWA weights $w_k = \sum_{i=m-k+1}^m w'_i$. Suppose that $\bar{x}$ is not efficient to the multiple criteria optimization problem (1). This means there exists a decision vector $x \in Q$ such that $f_i(\mathbf{x}) \geq f_i(\bar{\mathbf{x}})$ for all $i \in I$ and $f_{i_o}(\mathbf{x}) > f_{i_o}(\bar{\mathbf{x}})$ for some outcome index $i_o \in I$. Let us define $a_i = s_i(f_i(\mathbf{x}))$ according to formula (3). The pair $(\mathbf{x}, \mathbf{a})$ is then a feasible solution of problem (7). Moreover, $a_i \geq \bar{a}_i$ for all $i \in I$, where at least one strict inequality $a_{i_0} > \bar{a}_{i_0}$ holds. Hence, due to the strict monotonicity of the OWA aggregation with positive weighting vectors (Llamazares 2004), one gets $A_{\mathbf{w}}(\mathbf{a}) > A_{\mathbf{w}}(\bar{\mathbf{a}})$, which contradicts the optimality of $(\bar{\mathbf{x}}, \bar{\mathbf{a}})$ for problem (7). □

The following theorem shows that, for each properly efficient solution $\bar{\mathbf{x}}$ with tradeoffs bounded by appropriate Δ, there exist aspiration and reservation vectors such that $\bar{\mathbf{x}}$ with the corresponding values of the achievements is an optimal solution of problem (10), thus justifying the controllability of the interactive process by the reference levels.

Theorem 2 *For any positive and strictly increasing OWA weights* $0 < w_1 < w_2 < \cdots < w_m$, *if* $\bar{\mathbf{x}} \in Q$ *is a properly efficient solution with tradeoffs bounded by* $\Delta = \gamma w_m / \sum_{i=1}^{m-1} w_i$, *then there exist aspirations levels* r_i^a *and reservation levels* r_i^r *such that* $\bar{\mathbf{x}}$, *together with appropriate* $\bar{\mathbf{a}}, \bar{\mathbf{y}}, \bar{\mathbf{z}}, \bar{\mathbf{t}}, \bar{\mathbf{d}}$, *is an optimal solution of the corresponding problem* (10).

Proof Let $\bar{\mathbf{x}}$ with the corresponding outcome vector $\bar{\mathbf{y}}$ be a properly efficient solution with tradeoffs bounded by Δ. Let us set the reference levels as $r_i^r = \bar{y}_i$ and $r_i^a = \bar{y}_i + 1$ for $i \in I$ and define achievements $\bar{a}_i = s_i(\bar{y}_i)$ according to formula (3). We will show that $(\bar{\mathbf{x}}, \bar{\mathbf{a}})$ is an optimal solution of the corresponding OWA RPM problem (7). Suppose that there exists a feasible vector $\mathbf{x} \in Q$ with outcomes $\mathbf{y}$ such that for its achievements $a_i = s_i(f_i(\mathbf{x}))$, $i = 1, 2, \ldots, m$, one gets a better scalarizing achievement value $\sum_{i=1}^m w_i \theta_i(\mathbf{a}) > \sum_{i=1}^m w_i \theta_i(\bar{\mathbf{a}})$. Note that $\bar{a}_i = 0$ for all $i \in I$. Hence, $\theta_i(\mathbf{a}) - \theta_i(\bar{\mathbf{a}}) = a_{\tau(i)} - \bar{a}_{\tau(i)}$ for all $i \in I$, where τ is the ordering permutation for the achievements vector $\mathbf{a}$. Moreover, due to efficiency of $\bar{\mathbf{x}}$, $0 > a_{\tau(m)} - \bar{a}_{\tau(m)} \geq \gamma(y_{\tau(m)} - \bar{y}_{\tau(m)})$, and due to formula (3), $a_{\tau(i)} - \bar{a}_{\tau(i)} \leq y_{\tau(i)} - \bar{y}_{\tau(i)}$ whenever $y_{\tau(i)} - \bar{y}_{\tau(i)} > 0$. Thus, taking advantages of the proper efficiency inequalities (6) for $k = \tau(m)$, one gets

$$\sum_{i=1}^{m-1} w_i (y_{\tau(i)} - \bar{y}_{\tau(i)}) \leq - \sum_{i=1}^{m-1} w_i \Delta (y_{\tau(m)} - \bar{y}_{\tau(m)}) \leq -w_m \gamma (y_{\tau(m)} - \bar{y}_{\tau(m)})$$

which contradicts the inequality $\sum_{i=1}^m w_i \theta_i(\mathbf{a}) > \sum_{i=1}^m w_i \theta_i(\bar{\mathbf{a}})$ and thereby confirms the optimality of $(\bar{\mathbf{x}}, \bar{\mathbf{a}})$ for the corresponding OWA RPM problem (7).

Since, following formulas (4) and (9), problem (10) is equivalent to the OWA RPM problem (7) with partial achievement functions s_i defined by (3), the optimal solution $(\bar{\mathbf{x}}, \bar{\mathbf{a}})$ with outcomes $\bar{\mathbf{y}}$ can be expanded with appropriate values of $\bar{\mathbf{z}}, \bar{\mathbf{t}}, \bar{\mathbf{d}}$ to an optimal solution of problem (10). Actually, one needs to introduce (Ogryczak and Tamir 2003): $\bar{t}_k = \theta_k(\bar{\mathbf{a}})$ for $k \in I$, $\bar{d}_{ik} = \max\{\bar{t}_k - \bar{a}_i, 0\}$ for $i, k \in I$, and $\bar{z}_k = k\bar{t}_k - \sum_{i=1}^m \bar{d}_{ik}$ for $k \in I$. □

The OWA RPM while taking into account all the ordered achievements tends to find an efficient solution with possibly equal partial achievements. The following theorem shows that for any reference levels $r_i^a > r_i^r$ and any positive weights $\mathbf{w}'$, if there exists a properly efficient solution $\bar{\mathbf{x}}$ with tradeoffs bounded by $\Delta = w_m/(\sum_{i=1}^{m-1} w_i)$ generating equal achievements $\bar{a}_1 = \bar{a}_2 = \cdots = \bar{a}_m$, then $\bar{\mathbf{x}}$ with the corresponding values of auxiliary variables is an optimal solution of problem (10).

Theorem 3 *For any reference levels $r_i^a > r_i^r$, any positive and strictly increasing OWA weights $0 < w_1 < w_2 < \cdots < w_m$, if $\bar{\mathbf{x}} \in Q$ is a properly efficient solution with tradeoffs bounded by $\Delta = w_m/(\sum_{i=1}^{m-1} w_i)$ such that all its partial achievements are perfectly equal $\bar{a}_1 = \bar{a}_2 = \cdots = \bar{a}_m$, then $\bar{\mathbf{x}}$, together with appropriate $\bar{\mathbf{a}}, \bar{\mathbf{y}}, \bar{\mathbf{z}}, \bar{\mathbf{t}}, \bar{\mathbf{d}}$, is an optimal solution of the corresponding problem* (10).

Proof Let $\bar{\mathbf{x}}$ with the corresponding outcome vector $\bar{\mathbf{y}}$ be a properly efficient solution with tradeoffs bounded by Δ such that achievements $\bar{a}_i = s_i(\bar{y}_i)$ according to formula (3) are equal, i.e., $\bar{a}_i = \delta$ for all $i \in I$ with some constant δ. We will show that $(\bar{\mathbf{x}}, \bar{\mathbf{a}})$ is an optimal solution of the corresponding OWA RPM problem (7). Suppose that there exists a feasible vector $\mathbf{x} \in Q$ with outcomes $\mathbf{y}$ such that for its achievements $a_i = s_i(f_i(\mathbf{x})), i = 1, 2, \ldots, m$, one gets a better scalarizing achievement value $\sum_{i=1}^m w_i\theta_i(\mathbf{a}) > \sum_{i=1}^m w_i\theta_i(\bar{\mathbf{a}})$. Note that $\theta_i(\mathbf{a}) - \theta_i(\bar{\mathbf{a}}) = a_{\tau(i)} - \bar{a}_{\tau(i)}$ for all $i \in I$, where τ is the ordering permutation for the achievements vector $\mathbf{a}$. Moreover, due to efficiency of $\bar{\mathbf{x}}$, $0 > a_{\tau(m)} - \bar{a}_{\tau(m)} \geq \sigma(y_{\tau(m)} - \bar{y}_{\tau(m)})$, and due to formula (3), $a_{\tau(i)} - \bar{a}_{\tau(i)} \leq \sigma(y_{\tau(i)} - \bar{y}_{\tau(i)})$ whenever $y_{\tau(i)} - \bar{y}_{\tau(i)} > 0$, where $\sigma = \alpha$ for $\delta > 1$, $\sigma = \gamma$ for $\delta < 0$, and $\sigma = 1$ otherwise. Thus, taking advantages of the proper efficiency inequalities (6) for $k = \tau(m)$, one gets

$$\sum_{i=1}^{m-1} w_i\sigma(y_{\tau(i)} - \bar{y}_{\tau(i)}) \leq -\sum_{i=1}^{m-1} w_i\sigma\Delta(y_{\tau(m)} - \bar{y}_{\tau(m)}) = -w_m\sigma(y_{\tau(m)} - \bar{y}_{\tau(m)})$$

which contradicts the inequality $\sum_{i=1}^m w_i\theta_i(\mathbf{a}) > \sum_{i=1}^m w_i\theta_i(\bar{\mathbf{a}})$ and thereby confirms the optimality of $(\bar{\mathbf{x}}, \bar{\mathbf{a}})$ for the corresponding OWA RPM problem (7).

Since problem (10) is equivalent to the OWA RPM problem (7) with partial achievement functions s_i defined by (3), the optimal solution $(\bar{\mathbf{x}}, \bar{\mathbf{a}})$ with outcomes $\bar{\mathbf{y}}$ can be expanded with appropriate values of $\bar{\mathbf{z}}, \bar{\mathbf{t}}, \bar{\mathbf{d}}$ to an optimal solution of problem (10). Actually, one needs to introduce (Ogryczak and Tamir 2003): $\bar{t}_k = \theta_k(\bar{\mathbf{a}})$ for $k \in I$, $\bar{d}_{ik} = \max\{\bar{t}_k - \bar{a}_i, 0\}$ for $i, k \in I$, and $\bar{z}_k = k\bar{t}_k - \sum_{i=1}^m \bar{d}_{ik}$ for $k \in I$. □

When there does not exist any properly efficient solution with perfectly equal partial achievements, then the OWA RPM model generates another efficient solution still providing equitability of partial achievements with respect to the Pigou–Dalton principle of transfers (Kostreva et al. 2004). That means, a transfer of small amount from an individual achievement to any relatively worse-off individual achievement results in a more preferred achievement vector, i.e., whenever $a_i < a_j$ and $0 < \varepsilon \leq a_j - a_i$, then $\mathbf{a} + \varepsilon\mathbf{e}_i - \varepsilon\mathbf{e}_j$ is strictly preferred to $\mathbf{a}$, where $\mathbf{e}_i$ denotes the ith unit vector. Recall that this equitability property applies to uniformly measured partial achievements, and it does not enforce any equitability of the original outcomes.

3 Importance weighting of partial achievements

The standard RPM models allow weighting of several achievements only by straightforward rescaling of the achievement values (Ruiz et al. 2009). However, there are many decision situations where there is need for importance weighting of achievements. This can be related to the involvement of multiple DMs to specify the preferences like in procurement (Teich et al. 2006) or in more complex policy decisions. In public policy decisions the interests of multiple stakeholders have to be taken into account. Even when the decision is ultimately made by one individual, a politician, that individual has to consider the interests of different parties (Miettinen et al. 2008 and references therein). For instance, multicriteria analysis was needed for exploring stakeholders preferences for diverse future energy technologies developed with the European Integrated Project NEEDS (Makowski et al. 2009). Such Meta-criteria, which evaluate the distribution of results across several criteria, occur not only in multiperson decisions. The hierarchical importance of achievements can be introduced into RPM models by partitioning the set of achievements (criteria) into multiple preemptive optimization levels (Ogryczak 1997), while additional goal programming techniques applied to model requirements regarding relevant achievement function were considered in the so-called Meta-goal programming approach (Rodriguez et al. 2002). The OWA RPM model enables us to introduce importance weights to affect an achievement (criterion) importance by rescaling accordingly its measure within the distribution of all achievements (criteria). This concept (Ogryczak and Kozłowski 2008) can be formalized with the so-called Weighted OWA (WOWA) aggregations (Torra 1997; Torra and Narukawa 2007; Liu 2006) of the partial achievements.

Let $\mathbf{w} = (w_1, \ldots, w_m)$ and $\mathbf{p} = (p_1, \ldots, p_m)$ be weighting vectors of dimension m such that $w_i \geq 0$ and $p_i \geq 0$ for $i = 1, 2, \ldots, m$ and $\sum_{i=1}^m p_i = 1$ and $\sum_{i=1}^m w_i = 1$. The corresponding Weighted OWA aggregation of outcomes $\mathbf{a} = (a_1, \ldots, a_m)$ is originally (Torra 1997) defined as follows:

$$A_{\mathbf{w},\mathbf{p}}(\mathbf{a}) = \sum_{i=1}^{m} v_i(\mathbf{a})\theta_i(\mathbf{a}), \quad \text{where } v_i(\mathbf{a}) = V\left(\sum_{k\leq i} p_{\tau(k)}\right) - V\left(\sum_{k<i} p_{\tau(k)}\right) \tag{11}$$

with V a monotone increasing function that interpolates points $(\frac{i}{m}, \sum_{k\leq i} w_k)$ for $i = 1, 2, \ldots, m$ together with the point $(0, 0)$ and τ representing the ordering permutation for $\mathbf{a}$ (i.e., $a_{\tau(i)} = \theta_i(\mathbf{a})$). We use linear interpolation leading to the piecewise linear function V which is a straight line in the case of equal preferential weights ($w_i = 1/m$ for $i = 1, 2, \ldots, m$), thus allowing the WOWA to cover the standard weighted mean as a special case.

Consider, for instance, two achievements vectors $\mathbf{a}' = (0, 1)$ and $\mathbf{a}'' = (1, 0)$. While introducing preferential weights $\mathbf{w} = (0.1, 0.9)$, one may calculate the OWA averages: $A_{\mathbf{w}}(\mathbf{a}') = A_{\mathbf{w}}(\mathbf{a}'') = 0.1 \cdot 1 + 0.9 \cdot 0 = 0.1$. Further, let us introduce importance weights $\mathbf{p} = (0.75, 0.25)$ to express that results under the first achievement are three times more important than those related to the second criterion. To take into account the importance weights in the WOWA aggregation (11), we introduce the piecewise linear function

$$V(\xi) = \begin{cases} 0.1\xi/0.5 & \text{for } 0 \le \xi \le 0.5, \\ 0.1 + 0.9(\xi - 0.5)/0.5 & \text{for } 0.5 < \xi \le 1.0 \end{cases}$$

and calculate weights v_i according to formula (11) as V increments corresponding to importance weights of the ordered outcomes. In particular, for vector $\mathbf{a}'$, one gets $v_1(\mathbf{a}') = V(p_2) = 0.05$ and $v_2(\mathbf{a}') = 1 - V(p_2) = 0.95$, while $v_1(\mathbf{a}'') = V(p_1) = 0.55$ and $v_2(\mathbf{a}'') = 1 - V(p_1) = 0.45$. Finally, $A_{\mathbf{w},\mathbf{p}}(\mathbf{a}') = 0.05 \cdot 1 + 0.95 \cdot 0 = 0.05$ and $A_{\mathbf{w},\mathbf{p}}(\mathbf{a}'') = 0.55 \cdot 1 + 0.45 \cdot 0 = 0.55$.

One may alternatively compute the WOWA values by using the importance weights to replicate corresponding achievements and then calculate OWA aggregations. In the case of our importance weights $\mathbf{p}$ we need to consider three copies of achievement a_1 and one copy of achievement a_2, thus generating vectors $\tilde{\mathbf{a}}' = (0, 0, 0, 1)$ and $\tilde{\mathbf{a}}'' = (1, 1, 1, 0)$ of four equally important achievements. Original preferential weights must be then applied respectively to the average of the two smallest outcomes and the average of two largest outcomes. Indeed, we get $A_{\mathbf{w},\mathbf{p}}(\mathbf{a}') = 0.1 \cdot 0.5 + 0.9 \cdot 0 = 0.05$ and $A_{\mathbf{w},\mathbf{p}}(\mathbf{a}'') = 0.1 \cdot 1 + 0.9 \cdot 0.5 = 0.55$. This approach to WOWA calculation can easily be formalized for the case of a uniform quantile grid $\beta_j = j/J$, $j = 0, 1, \ldots, J$, enabling both the preferential and the importance weighting systems in the sense that there exists a positive integer l such that $J = ml$ and there exist positive integers $m_i = p_i J$ for $i = 1, 2, \ldots, m$. Then

$$A_{\mathbf{w},\mathbf{p}}(\mathbf{a}) = \sum_{i=1}^{m} \frac{w_i}{l} \sum_{j=(i-1)l+1}^{il} \theta_{k(\beta_j)}(\mathbf{a}), \tag{12}$$

where $k(\beta_j) = k$ for $\sum_{i<k} p_{\tau(i)} < \beta_j \le \sum_{i\le k} p_{\tau(i)}$, $i = 1, 2, \ldots, m$, with τ representing the ordering permutation for $\mathbf{a}$ (i.e., $a_{\tau(i)} = \theta_i(\mathbf{a})$).

Note that formula (12) allows one to express the WOWA aggregation with original preferential (OWA) weights w_i applied to averages of the corresponding portions of ordered achievements. Actually, this is a special case of the general formula expressing the WOWA as the weighted combination of averages within the quantile intervals according to the distribution defined by importance weights p_i (Ogryczak and Śliwiński 2007, 2009):

$$A_{\mathbf{w},\mathbf{p}}(\mathbf{a}) = \sum_{i=1}^{m} w_i m \int_{\frac{i-1}{m}}^{\frac{i}{m}} \overline{F}_{\mathbf{a}}^{(-1)}(\xi)\, d\xi, \tag{13}$$

where $\overline{F}_{\mathbf{a}}^{(-1)}$ is the (decreasing) quantile function for the distribution defined by importance weights p_i. It can also be mathematically formalized as follows. First, we introduce the right-continuous cumulative distribution function (cdf)

$$F_{\mathbf{a}}(d) = \sum_{i=1}^{m} p_i \delta_i(d), \tag{14}$$

where $\delta_i(d) = 1$ if $a_i \leq d$ and 0 otherwise. Next, we introduce the left-continuous inverse of the cumulative distribution function $F_{\mathbf{a}}$, ie., $F_{\mathbf{a}}^{(-1)}(\xi) = \inf\{\eta : F_{\mathbf{a}}(\eta) \geq \xi\}$ for $0 < \xi \leq 1$ and finally the quantile function $\overline{F}_{\mathbf{a}}^{(-1)}(\xi) = F_{\mathbf{a}}^{(-1)}(1-\xi)$.

The WOWA RPM can be formulated as based on the following optimization problem:

$$\max\left\{A_{\mathbf{w},\mathbf{p}}(\mathbf{a}) : \mathbf{x} \in Q, a_i = s_i\left(f_i(\mathbf{x})\right) \forall i \in I\right\} \tag{15}$$

used to generate current solutions according to the specified preferences. Such WOWA RPM model depends on two sets of weights **w** and **p** and obviously uses the reference levels. Let us recall that it is an extension of the OWA RPM with additional capability to introduce the importance weights **p**. Hence, the OWA weights w_i are the RPM internal parameters rather than the control parameters to be set by the DM during the interactive process. Geometric increasing series can be used to define the OWA RPM weights w_i, thus replacing the regularization parameter ε in the standard scalarizing achievement function (2). The importance weights **p** are designed to represent the preference information. Actually, depending on the decision situation, they may represent either a priori preference information not changed during the interactive analysis based then solely on the reference levels, or they can be used as additional control parameters supporting the interactive analysis. Note that in many applications, especially related to committee decision makers, the ability to use the importance weights may remarkably ease the organization of the interactive analysis.

Let us recall the RPM applied to the example of seven alternatives as given in Table 1. Applying importance weighting $\mathbf{p} = (\frac{4}{12}, \frac{3}{12}, \frac{2}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12})$ to solution achievements from Table 1 and using them together with the OWA weights **w** from Table 2, one gets the WOWA aggregations from Table 3. The corresponding RPM method then selects solution S6, similarly to the case of equal importance weights. On the other hand, when increasing the importance of the last outcome achievements with $\mathbf{p} = (\frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{7}{12})$, one gets the WOWA values from Table 4.

Formula (13) defines the WOWA value applying preferential weights w_i to importance weighted averages within quantile intervals. It may be reformulated with the tail averages:

$$A_{\mathbf{w},\mathbf{p}}(\mathbf{a}) = \sum_{k=1}^{m} w'_k m L\left(\mathbf{a}, \mathbf{p}, \frac{k}{m}\right), \quad \text{where } L(\mathbf{y}, \mathbf{p}, \xi) = \int_0^{\xi} F_{\mathbf{y}}^{(-1)}(\alpha)\, d\alpha \tag{16}$$

with weights $w'_k = w_{m-k+1} - w_{m-k}$ for $k = 1, \ldots, m-1$ and $w'_m = w_1$. The functions $L(\mathbf{y}, \mathbf{p}, \xi)$ defined by left-tail integrating of $F_{\mathbf{y}}^{(-1)}$ take the form of convex piecewise linear curves, the so-called absolute Lorenz curves (Ogryczak and Ruszczyński 2002) connected to the relation of the second-order stochastic dominance (SSD).

Table 3 WOWA RPM with importance weighting $\mathbf{p} = (\frac{4}{12}, \frac{3}{12}, \frac{2}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12})$ applied to solution achievements from Table 1 according to formula (12) with $J = 12$ and $l = 2$

	$\theta_{k(\frac{1}{12})}$	$\theta_{k(\frac{2}{12})}$	$\theta_{k(\frac{3}{12})}$	$\theta_{k(\frac{4}{12})}$	$\theta_{k(\frac{5}{12})}$	$\theta_{k(\frac{6}{12})}$	$\theta_{k(\frac{7}{12})}$	$\theta_{k(\frac{8}{12})}$	$\theta_{k(\frac{9}{12})}$	$\theta_{k(\frac{10}{12})}$	$\theta_{k(\frac{11}{12})}$	$\theta_{k(\frac{12}{12})}$	$A_{\mathbf{w},\mathbf{p}}$
S1	1.2	1.2	1.2	1.2	1.2	1.2	1.2	0.3	0.3	0.3	0.3	0.3	0.458
S2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	0.3	0.3	0.3	0.3	0.525
S3	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	0.3	0.3	0.3	0.638
S4	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	0.3	0.3	0.750
S5	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	0.3	0.3	0.750
S6	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.3	0.825
S7	1.0	1.0	0.6	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.319
w	0.02		0.03		0.05		0.15		0.25		0.5		

Table 4 WOWA RPM with importance weighting $\mathbf{p} = (\frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{7}{12})$ applied to solution achievements from Table 1 according to formula (12) with $J = 12$ and $l = 2$

	$\theta_{k(\frac{1}{12})}$	$\theta_{k(\frac{2}{12})}$	$\theta_{k(\frac{3}{12})}$	$\theta_{k(\frac{4}{12})}$	$\theta_{k(\frac{5}{12})}$	$\theta_{k(\frac{6}{12})}$	$\theta_{k(\frac{7}{12})}$	$\theta_{k(\frac{8}{12})}$	$\theta_{k(\frac{9}{12})}$	$\theta_{k(\frac{10}{12})}$	$\theta_{k(\frac{11}{12})}$	$\theta_{k(\frac{12}{12})}$	$A_{\mathbf{w},\mathbf{p}}$
S1	1.2	1.2	1.2	1.2	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.345
S2	1.2	1.2	1.2	1.2	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.345
S3	1.2	1.2	1.2	1.2	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.345
S4	1.2	1.2	1.2	1.2	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.345
S5	1.2	1.2	1.2	1.2	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.345
S6	1.0	1.0	1.0	1.0	1.0	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.353
S7	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.6	0.3	0.3	0.3	0.513
w	0.02		0.03		0.05		0.15		0.25		0.5		

According to (16), values of the function $L(\mathbf{a}, \mathbf{p}, \xi)$ for any $0 \le \xi \le 1$ can be given by optimization:

$$L(\mathbf{a}, \mathbf{p}, \xi) = \min_{u_i} \left\{ \sum_{i=1}^{m} a_i u_i : \sum_{i=1}^{m} u_i = \xi,\ 0 \le u_i \le p_i\ \forall i \right\}. \tag{17}$$

Introducing the dual variable t corresponding to the equation $\sum_{i=1}^{m} u_i = \xi$ and the variables d_i corresponding to upper bounds on u_i, one gets the following LP dual expression of $L(\mathbf{a}, \mathbf{p}, \xi)$:

$$L(\mathbf{a}, \mathbf{p}, \xi) = \max_{t, d_i} \left\{ \xi t - \sum_{i=1}^{m} p_i d_i : t - d_i \le a_i, d_i \ge 0\ \forall i \right\}. \tag{18}$$

Following (16) and (18) and taking into account piecewise linear partial achievement functions (3), one gets finally the following model for the WOWA Reference

Point Method with piecewise linear partial achievement functions (3):

$$
\begin{aligned}
\max \quad & \sum_{k=1}^{m} w'_k z_k \\
\text{s.t.} \quad & z_k = kt_k - m\sum_{i=1}^{m} p_i d_{ik}, && \forall k \in I, \\
& \mathbf{x} \in Q, \qquad y_i = f_i(\mathbf{x}), && \forall i \in I, \\
& a_i \geq t_k - d_{ik}, \qquad d_{ik} \geq 0, && \forall i,k \in I, \\
& a_i \leq \gamma\big(y_i - r_i^r\big)/\big(r_i^a - r_i^r\big), && \forall i \in I, \\
& a_i \leq \big(y_i - r_i^r\big)/\big(r_i^a - r_i^r\big), && \forall i \in I, \\
& a_i \leq \alpha\big(y_i - r_i^a\big)/\big(r_i^a - r_i^r\big) + 1, && \forall i \in I.
\end{aligned}
\tag{19}
$$

Theorem 4 *For any reference levels $r_i^a > r_i^r$ and any positive weights* $\mathbf{w}$ *and* $\mathbf{p}$, *if* $\bar{\mathbf{x}} \in Q$, *together with* $\bar{\mathbf{a}}, \bar{\mathbf{y}}, \bar{\mathbf{z}}, \bar{\mathbf{t}}, \bar{\mathbf{d}}$, *is an optimal solution of problem* (19), *then* $\bar{\mathbf{x}}$ *is an efficient solution of the corresponding multiple criteria optimization problem* (1).

Proof Let $(\bar{\mathbf{x}}, \bar{\mathbf{a}}, \bar{\mathbf{y}}, \bar{\mathbf{z}}, \bar{\mathbf{t}}, \bar{\mathbf{d}})$ be an optimal solution of problem (19) with some positive weighting vectors $\mathbf{w}$ and $\mathbf{p}$. Following formulas (4) and (18), one may notice that problem (19) is equivalent to the WOWA RPM problem (15) with partial achievement functions s_i defined by (3). Suppose that $\bar{x}$ is not efficient to the multiple criteria optimization problem (1). This means that there exists a decision vector $x \in Q$ such that $f_i(\mathbf{x}) \geq f_i(\bar{\mathbf{x}})$ for all $i \in I$ and $f_{i_o}(\mathbf{x}) > f_{i_o}(\bar{\mathbf{x}})$ for some outcome index $i_o \in I$. Let us define $a_i = s_i(f_i(\mathbf{x}))$ according to formula (3). The pair $(\mathbf{x}, \mathbf{a})$ is then a feasible solution of problem (15). Moreover, $a_i \geq \bar{a}_i$ for all $i \in I$, where at least one strict inequality $a_{i_0} > \bar{a}_{i_0}$ holds. Then (Ogryczak and Ruszczyński 2002) $\overline{F}_{\mathbf{a}}^{(-1)}(\xi) \geq \overline{F}_{\bar{\mathbf{a}}}^{(-1)}(\xi)$ for any $0 \leq \xi \leq 1$, and simultaneously $\int_0^1 \overline{F}_{\mathbf{a}}^{(-1)}(\xi)\, d\xi = \sum_{i=1}^m p_i a_i > \sum_{i=1}^m p_i \bar{a}_i = \int_0^1 \overline{F}_{\bar{\mathbf{a}}}^{(-1)}(\xi)\, d\xi$. Hence, following formula (13), one gets $A_{\mathbf{w},\mathbf{p}}(\mathbf{a}) > A_{\mathbf{w},\mathbf{p}}(\bar{\mathbf{a}})$, which contradicts the optimality of $(\bar{\mathbf{x}}, \bar{\mathbf{a}})$ for problem (15). □

When considering controllability of the interactive process by both the reference levels and the importance weights, one may use uniform importance weights $p_i = 1/m$ for $i \in I$, thus taking advantages of the OWA RPM controllability. Following Theorem 2, one may easily justify that for each properly efficient solution $\bar{\mathbf{x}}$ with tradeoffs bounded by $\Delta = \gamma w_m / (\sum_{i=1}^{m-1} w_i)$, there exist importance weights and aspiration and reservation vectors such that $\bar{\mathbf{x}}$ with the corresponding values of the achievements is an optimal solution of problem (19).

Theorem 5 *For any positive and strictly increasing normalized preferential weights* $0 < w_1 < w_2 < \cdots < w_m$ *with* $\sum_{i=1}^m w_i = 1$, *if* $\bar{\mathbf{x}} \in Q$ *is a properly efficient solution with tradeoffs bounded by* $\Delta = \gamma w_m / (\sum_{i=1}^{m-1} w_i)$, *then there exist aspirations levels* r_i^a, *reservation levels* r_i^r, *and positive importance weights* p_i *such that* $\bar{\mathbf{x}}$, *to-*

gether with appropriate $\bar{\mathbf{a}}, \bar{\mathbf{y}}, \bar{\mathbf{z}}, \bar{\mathbf{t}}, \bar{\mathbf{d}}$, *is an optimal solution of the corresponding problem* (19).

Proof When assuming equal importance weights $p_i = 1/m$ for $i \in I$, the WOWA RPM model (19) becomes the corresponding OWA RPM model (10). Thus Theorem 2 guarantees the existence of aspirations levels r_i^a and reservation levels r_i^r such that $\bar{\mathbf{x}}$, together with appropriate $\bar{\mathbf{a}}, \bar{\mathbf{y}}, \bar{\mathbf{z}}, \bar{\mathbf{t}}, \bar{\mathbf{d}}$, is an optimal solution of the corresponding problem (19) with assumed importance weights. □

Certainly, while dealing with given importance weights p_i, it might be more difficult to identify a specific efficient solution. Nevertheless, as shown in the following theorem, for any positive importance weights, for any properly efficient solution $\bar{\mathbf{x}} \in Q$ with tradeoffs bounded by appropriate Δ, there exist aspiration and reservation levels such that $\bar{\mathbf{x}}$ with the corresponding values of the achievements is an optimal solution of problem (19).

Theorem 6 *For any positive and strictly increasing normalized preferential weights* $0 < w_1 < w_2 < \cdots < w_m$ *and any positive importance weights* p_i, *if* $\bar{\mathbf{x}} \in Q$ *is a properly efficient solution with tradeoffs bounded by*

$$\Delta = \frac{m\bar{p}\gamma w_m}{(1 - m\bar{p}w_m)}, \qquad \text{where } \bar{p} = \min_{i \in I} p_i,$$

then there exist aspirations levels r_i^a *and reservation levels* r_i^r *such that* $\bar{\mathbf{x}}$, *together with appropriate* $\bar{\mathbf{a}}, \bar{\mathbf{y}}, \bar{\mathbf{z}}, \bar{\mathbf{t}}, \bar{\mathbf{d}}$, *is an optimal solution of the corresponding problem* (19).

Proof Let $\bar{\mathbf{x}}$ with the corresponding outcome vector $\bar{\mathbf{y}}$ be a properly efficient solution with tradeoffs bounded by Δ. Let us set the reference levels as $r_i^r = \bar{y}_i$ and $r_i^a = \bar{y}_i + 1$ for $i \in I$ and define achievements $\bar{a}_i = s_i(\bar{y}_i)$ according to formula (3). We will show that $(\bar{\mathbf{x}}, \bar{\mathbf{a}})$ is an optimal solution of the corresponding WOWA RPM problem (15). Suppose that there exists a feasible vector $\mathbf{x} \in Q$ with outcomes $\mathbf{y}$ such that for its achievements $a_i = s_i(f_i(\mathbf{x}))$, $i = 1, 2, \ldots, m$, one gets a better scalarizing achievement value $A_{\mathbf{w},\mathbf{p}}(\mathbf{a}) > A_{\mathbf{w},\mathbf{p}}(\bar{\mathbf{a}})$. Note that $\bar{a}_i = 0$ for all $i \in I$. Hence, following formula (11), $A_{\mathbf{w},\mathbf{p}}(\mathbf{a}) - A_{\mathbf{w},\mathbf{p}}(\bar{\mathbf{a}}) = \sum_{i=1}^m v_i(\mathbf{a})\theta_i(\mathbf{a}) - \sum_{i=1}^m v_i(\bar{\mathbf{a}})\theta_i(\bar{\mathbf{a}}) = \sum_{i=1}^m v_i(\mathbf{a})(a_{\tau(i)} - \bar{a}_{\tau(i)})$ where τ is the ordering permutation for the achievements vector $\mathbf{a}$. Moreover, due to the efficiency of $\bar{\mathbf{x}}$, $0 > a_{\tau(m)} - \bar{a}_{\tau(m)} \geq \gamma(y_{\tau(m)} - \bar{y}_{\tau(m)})$, and due to formula (3), $a_{\tau(i)} - \bar{a}_{\tau(i)} \leq y_{\tau(i)} - \bar{y}_{\tau(i)}$ whenever $y_{\tau(i)} - \bar{y}_{\tau(i)} > 0$. Thus, taking advantages of the proper efficiency inequalities (6) for $k = \tau(m)$, one gets

$$\sum_{i=1}^{m-1} v_i(\mathbf{a})(y_{\tau(i)} - \bar{y}_{\tau(i)}) \leq -\sum_{i=1}^{m-1} v_i(\mathbf{a})\Delta(y_{\tau(m)} - \bar{y}_{\tau(m)}) \leq -v_m(\mathbf{a})\gamma(y_{\tau(m)} - \bar{y}_{\tau(m)})$$

which contradicts to the inequality $\sum_{i=1}^m w_i\theta_i(\mathbf{a}) > \sum_{i=1}^m w_i\theta_i(\bar{\mathbf{a}})$ and thereby confirms the optimality of $(\bar{\mathbf{x}}, \bar{\mathbf{a}})$ for the corresponding WOWA RPM problem (15).

Since, following formulas (4) and (18), problem (19) is equivalent to the WOWA RPM problem (15) with partial achievement functions s_i defined by (3), the optimal solution $(\bar{\mathbf{x}}, \bar{\mathbf{a}})$ with outcomes $\bar{\mathbf{y}}$ can be expanded with appropriate values of $\bar{\mathbf{z}}, \bar{\mathbf{t}}, \bar{\mathbf{d}}$ to an optimal solution of problem (19). Actually, one needs to introduce (Ogryczak and Ruszczyński 2002): $\bar{t}_k = \overline{F}_{\bar{\mathbf{a}}}^{(-1)}(k/m)$ for $k \in I$, $\bar{d}_{ik} = \max\{\bar{t}_k - \bar{a}_i, 0\}$ for $i, k \in I$, and $\bar{z}_k = k\bar{t}_k - m\sum_{i=1}^{m} p_i \bar{d}_{ik}$ for $k \in I$. □

Due to introducing importance weights, the WOWA RPM allows one to distinguish various individual achievements. Therefore, opposite to the OWA RPM, it does not generally provide any direct equitability of partial achievements, although it can be considered equitable with respect to the importance weighted achievements in the sense that $\mathbf{a} + \frac{\varepsilon}{p_i}\mathbf{e}_i - \frac{\varepsilon}{p_j}\mathbf{e}_j$ is strictly preferred to $\mathbf{a}$ whenever $a_i < a_j$ and $0 < \varepsilon < (a_j - a_i)\min\{p_i, p_j\}$ (Ogryczak 2009a).

4 Illustrative example

In order to illustrate the WOWA RPM approach, let us analyze a simplified multiple criteria problem of information system selection. The decision is based on six criteria related to the system reliability, processing efficiency, investment costs, installation time, operational costs, and warranty period. All these attributes may be viewed as criteria, either maximized or minimized. Table 5 presents all the criteria with their measures units and optimization directions. There are also given the aspiration and reservation levels for each criterion and the normalized importance weights ($\mathbf{p}$) for several achievements. Five candidate systems have been accepted for the final selection procedure. All they meet the minimal requirements defined by the reservation levels. Table 6 presents, for all the systems (columns), their criteria values y_i and the corresponding partial achievement values a_i. The latter have been computed according to the piecewise linear formula (3) with $\alpha = 0.1$.

Table 7 presents, for all the systems (rows), their partial achievement values ordered from the largest to the smallest taking into account replications according to the importance weights allowing for easy WOWA aggregation computations according to formula (12). One may notice that, except of system D, all the other systems have the same worst achievement value $\min_i a_i = 0.33$. Selection among systems A, B, C, and E depends only on the regularization of achievements aggregation used in

Table 5 Criteria and their reference levels for the sample system selection

	f_1	f_2	f_3	f_4	f_5	f_6
Attributes	Reliability	Efficiency	Invest. cost	Install. time	Oprnl. cost	Warranty period
Units	1–10	CAPS	mln. EUR	months	mln. EUR	years
Optimization	max	max	min	min	min	max
Reservation	8	50	2	12	1.25	0.5
Aspiration	10	200	0	6	0.5	2
Importance weights	0.25	0.25	0.083	0.083	0.25	0.083

Table 6 Criteria values y_i and partial achievements a_i for five systems

i	System A y_i	System A a_i	System B y_i	System B a_i	System C y_i	System C a_i	System D y_i	System D a_i	System E y_i	System E a_i	Importance weight
1	10	1.00	9	0.50	10	1.00	9	0.50	10	1.00	0.250
2	200	1.00	100	0.33	170	0.80	90	0.27	150	0.67	0.250
3	1	0.50	0.3	0.85	0.8	0.60	0.2	0.90	0.5	0.75	0.083
4	8	0.67	3	1.05	8	0.67	8	0.67	5	1.02	0.083
5	1	0.33	1	0.33	0.6	0.87	0.2	1.04	1	0.33	0.250
6	2	1.00	2	1.00	1	0.33	2	1.00	1.5	0.67	0.083

Table 7 WOWA RPM selection for the sample system selection with calculations according to formula (12) for $J = 12$ and $l = 2$

	$\theta_{k(\frac{1}{12})}$	$\theta_{k(\frac{2}{12})}$	$\theta_{k(\frac{3}{12})}$	$\theta_{k(\frac{4}{12})}$	$\theta_{k(\frac{5}{12})}$	$\theta_{k(\frac{6}{12})}$	$\theta_{k(\frac{7}{12})}$	$\theta_{k(\frac{8}{12})}$	$\theta_{k(\frac{9}{12})}$	$\theta_{k(\frac{10}{12})}$	$\theta_{k(\frac{11}{12})}$	$\theta_{k(\frac{12}{12})}$	$A_{\mathbf{w},\mathbf{p}}$
A	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.67	0.50	0.33	0.33	0.33	0.47
B	1.05	1.00	0.85	0.50	0.50	0.50	0.33	0.33	0.33	0.33	0.33	0.33	0.37
C	1.00	1.00	1.00	0.87	0.87	0.87	0.80	0.80	0.80	0.67	0.60	0.33	0.56
D	1.04	1.04	1.04	1.00	0.90	0.67	0.50	0.50	0.50	0.27	0.27	0.27	0.37
E	1.02	1.00	1.00	1.00	0.75	0.67	0.67	0.67	0.67	0.33	0.33	0.33	0.45
w	0.02		0.03		0.05		0.1		0.2		0.6		

the RPM approach. The WOWA RPM method, taking into account the importance weights together with the preferential weights $\mathbf{w} = (0.02, 0.03, 0.05, 0.1, 0.2, 0.6)$, points out system C as the best one. The standard RPM (2) will select system A as better than all the others. Certainly, the WOWA RPM selection will change dramatically when decreasing importance of criterion f_5 and increasing importance of f_6.

Conclusions

The reference point method is a very convenient technique for interactive analysis of the multiple criteria optimization problems. It provides the DM with a tool for an open analysis of the efficient frontier. The interactive analysis is navigated with the commonly accepted control parameters expressing reference levels for the individual objective functions. The partial achievement functions quantify the DM satisfaction from the individual outcomes with respect to the given reference levels. The final scalarizing function is built as the augmented max–min aggregation of partial achievements, which means that the worst individual achievement is essentially maximized, but the optimization process is additionally regularized with the term representing the average achievement. The regularization by the average achievement is easily implementable, but it may disturb the basic max–min aggregation. In order to avoid inconsistencies caused by the regularization, the max–min solution may be regularized by taking into account also the second worst achievement, the third worse, and so on, thus resulting in much better modeling of the reference levels concept (Kostreva et al. 2004). We have shown that it can be achieved by a direct use of the OWA aggregation of the partial achievements as the final scalarizing achievement function. The corresponding OWA RPM method preserves the crucial requirements with respect to the efficiency of the generated solution and the controllability of interaction analysis by the reference levels as the only control parameters. Similar earlier ordered regularizations of the RPM have been based on the lexicographic approaches, thus resulting in complicated RPM models (Ogryczak 2008, 2009b) and not offering any explicit scalarizing achievement function which could be directly interpreted as expressing utility to be maximized. Developed optimization methods for ordered averages (Ogryczak and Śliwiński 2003) allow one to implement the OWA RPM quite effectively. The OWA RPM model with concave piecewise linear partial achievement functions (typically used in the RPM) can be formulated by the original constraints and criteria with simple auxiliary linear inequalities thus allowing for an efficient implementation.

Further, the OWA RPM model enables us to introduce importance weights to affect achievements importance by rescaling accordingly their measure within the distribution of all achievements. This concept (Ogryczak and Kozłowski 2008) takes advantages of the so-called Weighted OWA (WOWA) aggregations (Torra and Narukawa 2007) of the partial achievements. The standard RPM models allow weighting of several achievements only by straightforward rescaling of the achievement values (Ruiz et al. 2009), whereas there are many decision situations with a clear need for importance weighting of achievements. This can be related to the involvement of multiple DMs to specify the preferences like in complex policy decisions involving many stakeholders. We have shown that the WOWA RPM model meets the crucial requirements with respect to the efficiency of the generated solution and the controllability of interaction analysis by the reference levels and the importance weights. Moreover, opposite to similar earlier weighted ordered regularizations of the RPM based on the lexicographic approaches (Ogryczak 2009b), the WOWA RPM provides explicitly an analytical scalarizing achievement function which can be directly interpreted as expressing utility. The recent progress in optimization methods for the WOWA aggregations (Ogryczak and Śliwiński 2009) allows one to implement the WOWA RPM quite effectively by taking advantages of piecewise linear expression of the cumulated ordered achievements. Actually, in the case of concave piecewise linear partial achievement functions, the resulting formulation extends the original constraints and criteria with simple linear inequalities, thus allowing for a quite efficient implementation.

Our analysis has been focused on the RPM approaches. The OWA and WOWA aggregation can be similarly applied to regularize other max–min approaches to multiple criteria decision support. In particular, they can be used within goal programming methodology to reach better modeling of preferences with respect to reaching the aspiration levels for multiple goals (Rodriguez et al. 2002).

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