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Conditional median as a robust solution concept for uncapacitated location problems

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Abstract While making location decisions, one intends to increase effects (reduce distances) for the service recipients (clients). A conventional optimization approach to location problems considers only the optimality of locational decisions for specific clients data. Real-world applications inevitably involve errors and uncertainties in the operating conditions, and thereby the resulting performance may be lower than expected. In particular, a distribution system design is very sensitive to the varying demands for goods, and the demand changes may deteriorate drastically the system efficiency when optimized for different demand structure. Several approaches have been developed to deal with uncertain or imprecise data. The approaches focused on the quality or on the variation (stability) of the solution for some data domains are considered robust. Frequently, uncertainty is represented by limits (intervals) on possible values of demand weights varying independently rather than by scenarios for all the weights simultaneously. In this paper we show that a solution concept of the conditional median can be used to optimize effectively such robust location problems. The conditional median is a generalization of the minimax solution concept extended to take into account the number of services (the portion of demand) related to the worst performances. Namely, for a specified portion of demand, we take into account the corresponding portion of the maximum results, and we consider their average as the worst conditional mean to be minimized. Similar to the standard minimax approach, the minimization of the worst conditional mean can be defined by a linear objective and a number of auxiliary linear inequalities.

Keywords Location · Multiple criteria · Efficiency · Robustness · Conditional median

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1 Introduction

While locating facilities to meet some service demand of a given set of clients, the median solution concept, depending on the minimization of the demands weighted mean outcome, is well suited for the system efficiency maximization. However, serious uncertainty regarding demand data may significantly lower efficiency of the selected location pattern. This causes a need for some robust solution concepts.

Several approaches have been developed to deal with uncertain or imprecise data in optimization problem. The approaches focused on the quality or on the variation (stability) of the solution for some data domains are considered robust. The notion of robustness applied to decision problems was first introduced by Gupta and Rosenhead (1968). Practical importance of the performance sensitivity against data uncertainty and errors has later attracted considerable attention to the search for robust solutions. Actually, as suggested by Roy (1998), the concept of robustness should be applied not only to solutions but, more generally, to various assertions and recommendations generated within a decision support process. The precise concept of robustness depends on the way the uncertain data domains and the quality or stability characteristics are introduced. Typically, in robust analysis one does not attribute any probability distribution to represent uncertainties. Data uncertainty is rather represented by nonattributed scenarios. Since one wishes to optimize results under each scenario, robust optimization might be in some sense viewed as a multiobjective optimization problem where objectives correspond to the scenarios. However, despite of many similarities of such robust optimization concepts to multiobjective models, there are also some significant differences (Hites et al. 2006). Actually, robust optimization is a problem of optimal distribution of objective values under several scenarios (Ogryczak 2002) rather than a standard multiobjective optimization model.

A conservative notion of robustness focusing on worst-case scenario results is widely accepted and the minimax optimization is commonly used to seek robust solutions. The worst-case scenario analysis can be applied either to the absolute values of objectives (the absolute robustness) or to the regret values (the deviational robustness) (Kouvelis and Yu 1997). The latter, when considered from the multiobjective perspective, represents a simplified reference point approach with the utopian (ideal) objective values for all the scenario used as aspiration levels. Recently, a more advanced concept of ordered weighted averaging was introduced into robust optimization (Perny et al. 2006), thus, allowing one to optimize combined performances under the worst-case scenario together with the performances under the second worst scenario, the third worst, and so on. Such an approach exploits better the entire distribution of objective vectors in search for robust solutions, and, more importantly, it introduces some tools for modeling robust preferences. Actually, while more sophisticated concepts of robust optimization are considered within the area of discrete programming models, only the absolute robustness is usually applied to the majority of decision and design problems.

This study is focused on the weighted uncapacitated location problem where the demands uncertainty represented by intervals of possible values of weights varying independently rather than by scenarios for all the weights simultaneously. The center solution concept defined by the standard minimax optimization is a robust solution concept related to uncertainty represented by unbounded weights perturbations. This is usually too restrictive, thus lowering the average system efficiency. The so-called conditional median solution concept depending on minimization of the mean in β portion of the worst outcomes (Ogryczak and Zawadzki 2002) offers a compromise between the center and the median solution concepts. It is usually less restrictive than the center solution while similarly implementable with auxiliary linear inequalities. We show that the conditional median allows us to model robust solutions with proportional limits on weights perturbations. Arbitrary limits on possible values of the demand weights varying independently leads to the worst-case mean optimization models with variable coefficients (weights). We show that the models can be viewed as generalized conditional median models and, similarly to the conditional median solution concept, implemented with auxiliary linear inequalities.

The paper is organized as follows. In the next section we recall the conditional median solution concept, providing a new proof of the computational model which remains applicable for more general problems related to the robust solution concepts. Section 3 contains the main results. We show that the robust solution for proportional upper limits on weights perturbations is the conditional β-median for an appropriate β value. For proportional upper and lower limits on weights perturbation, the robust solution may be expressed with problem optimization problem of appropriately combined the median and the conditional median criteria. Finally, a general robust solution for any arbitrary intervals of demand weights perturbations can be expressed with optimization problem very similar to the conditional β-median and thereby easily implementable with auxiliary linear inequalities.

2 Location concepts

The generic location problem that we consider may be stated as follows. There is given a set $I = \{1, 2, \ldots, m\}$ of m clients (service recipients). Each client is represented by a specific point in the geographical space, and a set Q of location patterns (location decisions) is given. For each client $i \in I$, a function $f_i(\mathbf{y})$ of the location pattern $\mathbf{y}$ is defined. This function, called the individual objective function, measures the outcome (effect) $z_i = f_i(\mathbf{y})$ of the location pattern for client i (Marsh and Schilling 1994; Ogryczak 1999). In the simplest problems an outcome usually expresses the distance. However, we emphasize to the reader that we do not restrict our considerations to the case of outcomes measured as distances. They can be measured (modeled) as travel time, travel costs, and, in a more subjective way, as relative travel costs (e.g., travel costs by clients' incomes) or ultimately as the levels of clients' dissatisfaction (individual disutility) of locations. Let us define the set of attainable outcomes

$$A = \left\{\mathbf{z} : z_i = f_i(\mathbf{y})\ \forall i, \mathbf{y} \in Q\right\}. \tag{1}$$

In typical formulations of location problems related to desirable facilities, a smaller value of the outcome (distance) means a better effect (higher service quality or client satisfaction). This remains valid for location of obnoxious facilities if the distances are replaced with their complements to some large number. Therefore, without loss of generality, we can assume that each individual outcome z_i is to be minimized.

Further, some additional client weights $w_i > 0$ are included into location model to represent the service demand. Integer weights can be interpreted as numbers of unweighted clients located at exactly the same place (with distances 0 among them). While discussing such solution concepts, we will use the normalized client weights

$$\bar{w}_i = w_i \Big/ \sum_{i=1}^{m} w_i \quad \text{for } i = 1, 2, \dots, m \tag{2}$$

rather than the original quantities w_i. Note that, in the case of unweighted problem (all $w_i = 1$), all the normalized weights are given as $\bar{w}_i = 1/m$.

In general, the demand weights may affect the outcome values (distances) for individual clients since individual objective functions $f_i(\mathbf{y})$ actually also depend on demand weights. Such a situation occurs, for instance, in the capacitated location problem where distance to the (allocated) service facility depends on the demand weights values versus the facility capacity as some clients might be allocated to some further facilities due to limited capacity. Our analysis is focused on the case where the demand weights do not affect directly outcome values for individual clients, or, in other words, all constraints of the attainable set A remain unchanged while varying the demand weights. This is guaranteed for uncapacitated location problems.

Example 1 As an example, one may consider a typical discrete location problem (Mirchandani and Francis 1990) where a set of m clients and a set of n potential locations for the facilities are given. Further, the number (or the maximal number) p of facilities to be located is given ($p \le n$). The main decisions to be made can be described with the binary variables y_j ($j = 1, 2, \dots, n$) equal to 1 if location j is to be used and equal to 0 otherwise. To meet the problem requirements, the decision variables y_j have to satisfy the following constraints:

$$\sum_{j=1}^{n} y_j = p, \quad y_j \in \{0, 1\}, \quad \text{for } j = 1, 2, \dots, n. \tag{3}$$

For most location problems, the feasible set has a more complex structure due to explicit consideration of allocation decisions. These decisions are usually modeled with the additional allocation variables x'_{ij} ($i = 1, 2, \dots, m$; $j = 1, 2, \dots, n$) equal to 1 if location j is used to service client i and equal to 0 otherwise. The allocation variables have to satisfy the following constraints:

$$\sum_{j=1}^{n} x'_{ij} = 1 \quad \text{for } i = 1, 2, \dots, m; \tag{4}$$

$$x'_{ij} \le y_j \quad \text{for } i = 1, 2, \dots, m \text{ and } j = 1, 2, \dots, n; \tag{5}$$

$$x'_{ij} \in \{0, 1\} \quad \text{for } i = 1, 2, \dots, m \text{ and } j = 1, 2, \dots, n. \tag{6}$$

Let $d_{ij} \geq 0$ ($i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$) express the distance between client i and location j (or other effect of allocation client i to location j). With the explicit use of the allocation variables and the corresponding constraints (4)–(5), the individual outcomes (objective functions) z_i can be written in the following linear form:

$$z_i = \sum_{j=1}^{n} d_{ij} x'_{ij} \quad \text{for } i = 1, 2, \ldots, m. \tag{7}$$

In the case of location of desirable facilities, a smaller value of the individual outcome means a better effect (smaller distance). This remains valid for location of obnoxious facilities if the distance coefficients are replaced with their complements to some large number: $d'_{ij} = d - d_{ij}$, where $d > d_{ij}$ for all $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Therefore, we can assume that each outcome z_i is to be minimized. The attainable set (1) may be considered as $A = \{\mathbf{z} = (z_1, z_2, \ldots, z_m) : (3)\text{–}(7)\}$.

We do not assume any special form of the feasible set while analyzing properties of the solution concepts. We rather allow the feasible set to be a general, possibly discrete (nonconvex), set. Similarly, we do not assume any special form of the individual objective functions nor their special properties (like convexity) while analyzing properties of the solution concepts. Therefore, the results of our analysis apply to various location problems.

A host of operational models has been developed to deal with facility location optimization (cf. Francis et al. 1992; Love et al. 1988; Malczewski and Ogryczak 1995, 1996; Mirchandani and Francis 1990; Nickel and Puerto 2005). Most classical location studies focus on the minimization of the mean (or total) distance (the median concept) or the minimization of the maximum distance (the center concept) to the service facilities (Morrill and Symons 1977). Both the median and the center solution concepts are well defined for weighted location models using client weights $w_i > 0$ to represent several clients (service demand) at the same geographical point. Exactly, for the weighted location problem, the median solution concept is defined by minimization of the objective function expressing the mean (average) outcome

$$\mu(\mathbf{z}) = \sum_{i=1}^{m} \bar{w}_i z_i,$$

but it is also equivalent to minimization of the total outcome $\sum_{i=1}^{m} w_i z_i$. The center solution concept is defined by minimization of the objective function representing the *maximum* (worst) outcome

$$M(\mathbf{z}) = \max_{i=1,\ldots,m} z_i,$$

and it is not affected by the client weights at all.

For unweighted location problems, a compromise solution concept was introduced by Slater (1978) as the so-called k-centrum where the sum of the k largest distances is minimized. Consistently with typical distribution characteristics, Peeters (1998) introduced four optimization criteria on outcomes (distances): upper (lower) k-median, where the sum of the k largest (smallest) outcomes was minimized, and upper (lower) k-center, where the kth largest (smallest) outcome itself was minimized. According to this classification, the k-centrum should be rather called the upper k-median. The k-centrum concept is restricted to unweighted problems. Although some weights are used to scale the specific distances (Tamir 2001) (which may be considered as a definition of distance dependent outcomes), the demand weights as defining the distribution of clients are not considered. Ogryczak and Zawadzki (2002) introduced a parametric generalization of the k-centrum concept applied to weighted problems by taking into account the portion of demand related to the largest outcomes (distances) rather than the specific number of worst outcomes. Namely, for a specified portion β of demand, the entire β portion (quantile) of the largest outcomes is taken into account, and their average is considered as the (worst) conditional β-mean outcome. Following the Peeters' classification (Peeters 1998), we call an (upper) conditional median every location pattern which minimizes the corresponding conditional mean outcome. According to this definition, the concept of conditional median is based on averaging restricted to the portion of the worst outcomes. For the unweighted location problems and $\beta = k/m$, the conditional β-mean represents the average of the k largest outcomes, thus modeling the k-centrum solution concept.

The conditional mean can be mathematically formalized as follows (Ogryczak 2002; Ogryczak and Ruszczyński 2002). First, we introduce the left-continuous right tail cumulative distribution function (cdf):

$$F_{\mathbf{z}}(d) = \sum_{i=1}^{m} \bar{w}_i \kappa_i(d), \quad \text{where } \kappa_i(d) = \begin{cases} 1 & \text{if } z_i \geq d, \\ 0 & \text{otherwise,} \end{cases} \tag{8}$$

which for any real (outcome) value d provides the measure of outcomes greater or equal to d. Next, we introduce the quantile function $F_{\mathbf{z}}^{(-1)}$ as the right-continuous inverse of the cumulative distribution function $F_{\mathbf{z}}$:

$$F_{\mathbf{z}}^{(-1)}(\beta) = \sup\{\eta : F_{\mathbf{z}}(\eta) \geq \beta\} \quad \text{for } 0 < \beta \leq 1.$$

By integrating $F_{\mathbf{z}}^{(-1)}$ one gets the (worst) conditional mean

$$\mu_\beta(\mathbf{z}) = \frac{1}{\beta} \int_0^\beta F_{\mathbf{z}}^{(-1)}(\alpha)\, d\alpha \quad \text{for } 0 < \beta \leq 1. \tag{9}$$

Minimization of the conditional β-mean

$$\min_{\mathbf{z} \in A} \mu_\beta(\mathbf{z}) \tag{10}$$

defines the conditional β-median solution concept. When parameter β approaches 0, the conditional β-mean tends to the largest outcome ($M(\mathbf{z}) = \lim_{\beta \to 0_+} \mu_\beta(\mathbf{z})$), and the conditional median becomes the center. On the other hand, for $\beta = 1$, the corresponding conditional mean becomes the standard mean ($\mu_1(\mathbf{z}) = \mu(\mathbf{z})$), and one gets the median location.

Note that, due to the finite distribution of outcomes z_i $(i = 1, 2, \ldots, m)$ in our location problems, the conditional β-mean is well defined by the following optimization:

$$\mu_\beta(\mathbf{z}) = \frac{1}{\beta} \max_{u_i} \left\{ \sum_{i=1}^{m} z_i u_i : \sum_{i=1}^{m} u_i = \beta,\ 0 \le u_i \le \bar{w}_i\ \forall i \right\}. \tag{11}$$

The above problem is a Linear Program (LP) for a given outcome vector $\mathbf{z}$, while it becomes nonlinear for $\mathbf{z}$ being a vector of variables as in the β-median problem (10). It turns out that this difficulty can be overcome by an equivalent LP formulation of the β-mean that allows one to implement the β-median problem (10) with auxiliary linear inequalities. Namely, the following theorem is valid (Ogryczak and Zawadzki 2002), although we introduce a new proof which can be further generalized for a family of robust location solution concepts we consider.

Theorem 1 *For any outcome vector* $\mathbf{y}$ *with the corresponding demand weights* w_i *and for any real value* $0 < \beta \le 1$, *the conditional* β*-mean outcome is given by the following linear program*:

$$\mu_\beta(\mathbf{z}) = \min_{t, d_i} \left\{ t + \frac{1}{\beta} \sum_{i=1}^{m} \bar{w}_i d_i : z_i \le t + d_i,\ d_i \ge 0\ \forall i \right\}. \tag{12}$$

Proof The theorem can be proven by taking advantage of the LP dual to (11). Introducing dual variable t corresponding to the equation $\sum_{i=1}^{m} u_i = \beta$ and variables d_i corresponding to upper bounds on u_i, one gets the LP dual (12). Due to the duality theory, for any given vector $\mathbf{z}$, the conditional β-mean $\mu_\beta(\mathbf{z})$ can be found as the optimal value of the LP problem (12). □

Following Theorem 1, the conditional β-median can be found as an optimal solution to the optimization problem

$$\min_{\mathbf{z}, \mathbf{d}, t} \left\{ t + \frac{1}{\beta} \sum_{i=1}^{m} \bar{w}_i d_i : \mathbf{z} \in A;\ z_i \le t + d_i,\ d_i \ge 0\ \forall i \right\} \tag{13}$$

or, in a more compact form,

$$\min_{\mathbf{z}, t} \left\{ t + \frac{1}{\beta} \sum_{i=1}^{m} \bar{w}_i (z_i - t)^+ : \mathbf{z} \in A \right\},$$

where $(.)^+$ denotes the nonnegative part of a number.

For the special case of an unweighted location problem ($w_i = 1/m$ for all $i \in I$) and $\beta = k/m$, one gets the conditional k/m-median. Model (13) takes then the form

$$\min_{\mathbf{z}, \mathbf{d}, t} \left\{ t + \frac{1}{k} \sum_{i=1}^{m} d_i : \mathbf{z} \in A;\ z_i \le t + d_i,\ d_i \ge 0\ \forall i \right\}, \tag{14}$$

which is the same as the computational formulation of the k-centrum introduced in Ogryczak and Tamir (2003). Hence, Theorem 1 and model (13) generalize the k-centrum formulation of Ogryczak and Tamir (2003) allowing one to consider demand weights and arbitrary size parameter β but preserving the simple structure and dimension of the optimization problem.

3 Robust locations

The median solution concept is usually very attractive solution concept due to maximizing the system efficiency taking into account the demands. The median solution is defined as

$$\min_{\mathbf{z}\in A}\left\{\sum_{i=1}^{m}\bar{w}_i z_i\right\}. \tag{15}$$

However, in practical problems the demand weights may vary. Therefore, a robust solution is sought which performs well under uncertain demand weights. Recall that our analysis is focused on the case where the demand weights perturbations do not affect directly outcome values for individual clients, or in other words, all constraints of the attainable set A remain unchanged while varying the demand weights. This is clearly guaranteed for uncapacitated location problems.

The simplest representation of uncertainty depends on a number of predefined scenarios $s = 1,\dots,r$. Let $\bar{w}_i^s$ denote the realization of demand i under scenario s. Then one may seek for a robust solution by minimizing the mean distance under the worst scenario

$$\min_{\mathbf{z}\in A}\max_{s=1,\dots,r}\left\{\sum_{i=1}^{m}\bar{w}_i^s z_i\right\} = \min_{\mathbf{z}\in A}\left\{\zeta : \zeta \geq \sum_{i=1}^{m}\bar{w}_i^s z_i \ \forall s\right\}$$

or by minimizing the maximum regret (Fernandez et al. 2001; Puerto et al. 2009)

$$\min_{\mathbf{z}\in A}\max_{s=1,\dots,r}\left\{\sum_{i=1}^{m}\bar{w}_i^s z_i - \bar{b}^s\right\} = \min_{\mathbf{z}\in A}\left\{\zeta : \zeta \geq \sum_{i=1}^{m}\bar{w}_i^s z_i - \bar{b}^s \ \forall s\right\},$$

where $\bar{b}^s$ represents the best value under scenario s,

$$\bar{b}^s = \min_{\mathbf{z}\in A}\left\{\sum_{i=1}^{m}\bar{w}_i^s z_i\right\}.$$

Frequently, uncertainty is represented by limits (intervals) on possible values of weights varying independently rather than by scenarios for all the weights simultaneously. We focus on such representation to define robust location concept. Assume that the demand weights $\bar{w}_i$ may be affected by perturbations varying within intervals $[-\delta_i, \Delta_i]$. Note that the weights are normalized, and, although varying independently, they must total to 1. Thus the demand weights belong to the hypercube:

$$\mathbf{u} \in W = \left\{(u_1, u_2, \ldots, u_m) : \sum_{i=1}^{m} u_i = 1,\ \bar{w}_i - \delta_i \le u_i \le \bar{w}_i + \Delta_i\ \forall i\right\}.$$

Alternatively one may consider completely independent perturbations of unnormalized weights w_i and normalize them later to define set W. Focusing on the mean outcome as the primary system efficiency measure to be optimized, we get the robust median solution concept

$$\min_{\mathbf{z}} \max_{\mathbf{u}} \left\{\sum_{i=1}^{m} u_i z_i : \mathbf{u} \in W,\ \mathbf{z} \in A\right\}. \tag{16}$$

Further, taking into account the assumption that all the constraints of attainable set A remain unchanged while the demand weights are perturbed, the robust median solution can be rewritten as

$$\min_{\mathbf{z}\in A} \max_{\mathbf{u}\in W} \sum_{i=1}^{m} u_i z_i = \min_{\mathbf{z}\in A} \left\{\max_{\mathbf{u}\in W} \sum_{i=1}^{m} u_i z_i\right\} = \min_{\mathbf{z}\in A} \mu^w(\mathbf{z}), \tag{17}$$

where

$$\begin{aligned}\mu^w(\mathbf{z}) &= \max_{\mathbf{u}\in W} \sum_{i=1}^{m} u_i z_i \\ &= \max_{u_i} \left\{\sum_{i=1}^{m} z_i u_i : \sum_{i=1}^{m} u_i = 1,\ \bar{w}_i - \delta_i \le u_i \le \bar{w}_i + \Delta_i\ \forall i\right\}\end{aligned} \tag{18}$$

represents the worst-case mean distance for given outcome vector $\mathbf{z} \in A$.

Note that in the case of $\delta_i = \Delta_i = 0$ (no perturbations/uncertainty at all) one gets the standard mean outcome $\mu^w(\mathbf{z}) = \sum_{i=1}^{m} z_i \bar{w}_i$ and thus the original median solution concept. On the other hand, for the case of unlimited perturbations ($\delta_i = \bar{w}_i$ and $\Delta_i = 1 - \bar{w}_i$), the worst-case mean outcome

$$\mu^w(\mathbf{z}) = \max_{u_i} \left\{\sum_{i=1}^{m} z_i u_i : \sum_{i=1}^{m} u_i = 1,\ 0 \le u_i \le 1\ \forall i\right\} = \max_{i=1,\ldots,m} z_i$$

becomes the worst outcome, thus leading to the center solution concept.

For the special case of proportional perturbation limits $\delta_i = \delta \bar{w}_i$ and $\Delta_i = \Delta \bar{w}_i$ with positive parameters δ and Δ, one gets

$$\mu^w(\mathbf{z}) = \max_{u_i} \left\{\sum_{i=1}^{m} z_i u_i : \sum_{i=1}^{m} u_i = 1,\ \bar{w}_i(1-\delta) \le u_i \le \bar{w}_i(1+\Delta)\ \forall i\right\}. \tag{19}$$

In particular, when lower limits are relaxed ($\delta = 1$), this becomes the classical conditional mean outcome (Ogryczak and Śliwiński 2002; Ogryczak and Zawadzki 2002) with $\beta = 1/(1+\Delta)$. Thus the conditional median represents the robust median solution concept for proportionally upper bounded perturbations.

Theorem 2 *The conditional β-median represents a concept of robust median solution* (17) *for proportionally upper bounded perturbations $\Delta_i = \Delta\bar{w}_i$ with $\Delta = (1-\beta)/\beta$ and relaxed lower ones $\delta_i = \bar{w}_i$ for all $i \in I$.*

Proof For proportionally bounded upper perturbations $\Delta_i = \Delta\bar{w}_i$ and $\delta_i = \bar{w}_i$, the corresponding worst-case mean distance (18) can be expressed as follows:

$$\begin{aligned}\mu^w(\mathbf{z}) &= \max_{u_i}\left\{\sum_{i=1}^m z_i u_i : \sum_{i=1}^m u_i = 1,\ 0 \le u_i \le \bar{w}_i(1+\Delta)\ \forall i\right\}\\ &= (1+\Delta)\max_{u'_i}\left\{\sum_{i=1}^m z_i u'_i : \sum_{i=1}^m u'_i = \frac{1}{1+\Delta},\ 0 \le u'_i \le \bar{w}_i\ \forall i\right\}\\ &= (1+\Delta)\mu_{\frac{1}{1+\Delta}}(\mathbf{z}).\end{aligned}$$

□

As the conditional median is easily defined by auxiliary LP constraints, the same applies to the robust median solution concept for proportionally bounded upper perturbations and relaxed lower ones.

Corollary 1 *The robust median solution concept* (17) *for proportionally bounded upper perturbations $\Delta_i = \Delta\bar{w}_i$ and relaxed lower limits $\delta_i = \bar{w}_i$ for all $i \in I$ can be found by simple expansion of the optimization problem with auxiliary linear constraints and variables to the following*:

$$\min_{\mathbf{z},\mathbf{d},t}\left\{t + (1+\Delta)\sum_{i=1}^m \bar{w}_i d_i : \mathbf{z} \in A;\ z_i \le t + d_i,\ d_i \ge 0\ \forall i\right\}. \tag{20}$$

In the general case of proportional perturbation limits (19) the robust median solution concepts cannot be directly expressed as an appropriate conditional β-median. It turns out, however, that it can be expressed by the optimization with combined criteria of the conditional β-median and the original median.

Theorem 3 *The robust median solution concept* (17) *for proportionally bounded perturbations $\Delta_i = \Delta\bar{w}_i$ and $\delta_i = \delta\bar{w}_i$ for all $i \in I$ is equivalent to the convex combination of the median and conditional β-median criteria minimization*

$$\min_{\mathbf{z}\in A}\mu^w(\mathbf{z}) = \min_{\mathbf{z}\in A}(1+\Delta)\left[\lambda\mu_\beta(\mathbf{z}) + (1-\lambda)\mu(\mathbf{z})\right] \tag{21}$$

with $\beta = \delta/(\Delta+\delta)$ and $\lambda = (\Delta+\delta)/(1+\Delta)$.

Proof For proportionally bounded perturbations $\Delta_i = \Delta \bar{w}_i$ and $\delta_i = \delta \bar{w}_i$ the corresponding worst-case mean distance (18) can be expressed as follows:

$$\begin{aligned}
\mu^w(\mathbf{z}) &= \max_{u_i} \left\{ \sum_{i=1}^m z_i u_i : \sum_{i=1}^m u_i = 1,\ \bar{w}_i(1-\delta) \le u_i \le \bar{w}_i(1+\Delta)\ \forall i \right\} \\
&= (1+\Delta) \max_{u_i'} \left\{ \sum_{i=1}^m z_i u_i' : \sum_{i=1}^m u_i' = \frac{1}{1+\Delta},\ \bar{w}_i \frac{1-\delta}{1+\Delta} \le u_i' \le \bar{w}_i\ \forall i \right\} \\
&= (1+\Delta) \max_{u_i''} \left\{ \sum_{i=1}^m z_i u_i'' : \sum_{i=1}^m u_i'' = \frac{\delta}{1+\Delta},\ 0 \le u_i'' \le \bar{w}_i \frac{\Delta+\delta}{1+\Delta}\ \forall i \right\} \\
&\quad + (1-\delta) \sum_{i=1}^m z_i \bar{w}_i \\
&= (\Delta+\delta) \max_{u_i'''} \left\{ \sum_{i=1}^m z_i u_i''' : \sum_{i=1}^m u_i''' = \frac{\delta}{\Delta+\delta},\ 0 \le u_i''' \le \bar{w}_i\ \forall i \right\} \\
&\quad + (1-\delta)\mu(\mathbf{z}) \\
&= (1+\Delta)\left[\frac{\Delta+\delta}{1+\Delta} \mu_{\frac{\delta}{\Delta+\delta}}(\mathbf{z}) + \frac{1-\delta}{1+\Delta}\mu(\mathbf{z}) \right],
\end{aligned}$$

which completes the proof. □

Following Theorems 1 and 3, the robust median solution concept (17) can be expressed as an LP expansion of the original median problem.

Corollary 2 *The robust median solution concept* (17) *for proportionally bounded perturbations $\Delta_i = \Delta \bar{w}_i$ and $\delta_i = \delta \bar{w}_i$ for all $i \in I$ can be found by simple expansion of the median problem with auxiliary linear constraints and variables to the following*:

$$\min_{\mathbf{z},\mathbf{d},t} \left\{ \sum_{i=1}^m \bar{w}_i z_i + \frac{\Delta+\delta}{1-\delta} t + \frac{(\Delta+\delta)^2}{\delta(1-\delta)} \sum_{i=1}^m \bar{w}_i d_i : \mathbf{z} \in A;\ z_i \le t + d_i,\ d_i \ge 0\ \forall i \right\}. \quad (22)$$

In the general case of arbitrary intervals of demand weights perturbations, the worst-case mean distance (18) cannot be expressed as a conditional β-mean or its combination. Nevertheless, the structure of optimization problem (18) remains very similar to that of the conditional β-mean (11). Note that problem (18) is an LP for a given outcome vector $\mathbf{z}$, while it becomes nonlinear for $\mathbf{z}$ being a vector of variables. This difficulty can be overcome similar to Theorem 1 for the conditional β-mean.

Theorem 4 *For any arbitrary intervals* $[-\delta_i, \Delta_i]$ (*for all* $i \in I$) *of demand weights perturbations*, *the corresponding worst-case mean outcome* (18) *can be given as*

$$\mu^w(\mathbf{z}) = \min_{t, d_i^u, d_i^l} \left\{ t + \sum_{i=1}^m (\bar{w}_i + \Delta_i) d_i^u - \sum_{i=1}^m (\bar{w}_i - \delta_i) d_i^l : \right.$$
$$\left. t + d_i^u - d_i^l \ge z_i, \ d_i^u, d_i^l \ge 0 \ \forall i \right\}. \tag{23}$$

Proof The theorem can be proven by taking advantages of the LP dual to (18). Introducing the dual variable t corresponding to the equation $\sum_{i=1}^m u_i = 1$ and the variables d_i^u and d_i^l corresponding to upper and lower bounds on u_i, respectively, one gets the following LP dual to problem (18). Due to the duality theory, for any given vector $\mathbf{z}$, the worst-case mean outcome $\mu^w(\mathbf{z})$ can be found as the optimal value of the LP problem (23). □

Following Theorem 4, the robust median solution concept (17) can be expressed similar to the conditional β-median with auxiliary linear inequalities expanding the original constraints.

Corollary 3 *For any arbitrary intervals* $[-\delta_i, \Delta_i]$ (*for all* $i \in I$) *of demand weights perturbations*, *the corresponding robust median solution* (17) *can be given by the following optimization problem*:

$$\min_{\mathbf{z}, t, d_i^u, d_i^l} \left\{ t + \sum_{i=1}^m (\bar{w}_i + \Delta_i) d_i^u - \sum_{i=1}^m (\bar{w}_i - \delta_i) d_i^l : \right.$$
$$\left. \mathbf{z} \in A; \ t + d_i^u - d_i^l \ge z_i, \ d_i^u, d_i^l \ge 0 \ \forall i \right\}. \tag{24}$$

Actually, there is a possibility to represent general robust median solution (17) with optimization problem even more similar to the conditional β-median and thereby with lower number of auxiliary variables than in (24).

Theorem 5 *For any arbitrary intervals* $[-\delta_i, \Delta_i]$ (*for all* $i \in I$) *of demand weights perturbations*, *the corresponding robust median solution* (17) *can be given by the following optimization problem*:

$$\min_{\mathbf{z}, t, d_i} \left\{ \sum_{i=1}^m (\bar{w}_i - \delta_i) z_i + \bar{\delta} t + \sum_{i=1}^m (\Delta_i + \delta_i) d_i : \mathbf{z} \in A; \ t + d_i \ge z_i, \ d_i \ge 0 \ \forall i \right\}, \tag{25}$$

where $\bar{\delta} = \sum_{i=1}^m \delta_i$.

Proof Note that the worst-case mean (18) may be transformed as follows:

$$\mu^w(\mathbf{z}) = \max_{u_i} \left\{ \sum_{i=1}^m z_i u_i : \sum_{i=1}^m u_i = 1,\ \bar{w}_i - \delta_i \le u_i \le \bar{w}_i + \Delta_i\ \forall i \right\}$$
$$= \max_{u'_i} \left\{ \sum_{i=1}^m z_i u'_i : \sum_{i=1}^m u'_i = \sum_{i=1}^m \delta_i,\ 0 \le u'_i \le \Delta_i + \delta_i\ \forall i \right\}$$
$$+ \sum_{i=1}^m z_i(\bar{w}_i - \delta_i). \tag{26}$$

Next, replacing the maximization over variables u_i with the corresponding dual, we get

$$\mu^w(\mathbf{z}) = \min_{t,d_i} \left\{ \left(\sum_{i=1}^m \delta_i\right) t + \sum_{i=1}^m (\Delta_i + \delta_i) d_i : t + d_i \ge z_i,\ d_i \ge 0\ \forall i \right\} + \sum_{i=1}^m (\bar{w}_i - \delta_i) z_i.$$

Further, minimization over $\mathbf{z} \in A$ leads us to formula (25), which completes the proof. □

For a special case of arbitrary upper bounds Δ_i and completely relaxed lower bound, we get the following result.

Corollary 4 *For any arbitrary upper bounds Δ_i and relaxed lower ones $\delta_i = \bar{w}_i$ (for all $i \in I$) on demand weights perturbations, the corresponding robust median solution* (17) *can be given by the following optimization problem*:

$$\min_{\mathbf{z},t,d_i} \left\{ t + \sum_{i=1}^m (\Delta_i + \bar{w}_i) d_i : \mathbf{z} \in A;\ t + d_i \ge z_i,\ d_i \ge 0\ \forall i \right\}. \tag{27}$$

Note that optimization problem (27) is very similar to the conditional β-median model (13). Indeed, in the case of proportional upper limits $\Delta_i = \Delta \bar{w}_i$ (for all $i \in I$) problem (27) simplifies to (20), as stated in Corollary 1.

Concluding remarks

For location problem with demand weights, the median solution concept is well suited for system efficiency maximization. However, real-life demand weights inevitably involve errors and uncertainties in the operating conditions, and thereby the resulting performance may be lower than expected. We have analyzed the robust median solution concept where demands uncertainty is represented by limits (intervals) on possible values of weights varying independently. Such an approach, in general, leads to complex optimization models with variable coefficients (weights). We have shown that in the case of uncapacitated location problem the robust median solution concepts can be expressed with auxiliary linear inequalities, similarly to the conditional

β-median solution concept (Ogryczak and Zawadzki 2002) based on minimization of the mean in β portion of the worst outcomes. Actually, the robust median solution for proportional upper limits on weights perturbations turns out to be the conditional β-median for an appropriate β value. For proportional upper and lower limits on weights perturbation, the robust median solution may be sought by optimization of appropriately combined the median and the conditional median criteria. Finally, a general robust median solution for any arbitrary intervals of demand weights perturbations can be expressed with an optimization problem very similar to the conditional β-median and thereby easily implementable with auxiliary linear inequalities.

The robust median optimization problems, similar to the standard minimax approach, may be modeled with a number of simple linear inequalities. As the conditional median problems, with the use of a simple general-purpose MIP code they usually need computational efforts larger than that for the median but smaller than that for the center (Ogryczak and Zawadzki 2002). Certainly, large-scale real-life location problems will require some specialized algorithms. Therefore, research on efficient specialized algorithms for various specific types of location problems should be continued.

Our analysis has shown that the robust median solution concept is closely related with the conditional median, which is the basic equitable solution concept (Kostreva and Ogryczak 1999b). It corresponds to recent approaches to the robust optimization based on the equitable optimization (Miettinen et al. 2008; Perny et al. 2006; Kostreva et al. 2004). Further study on equitable location concepts (Kostreva and Ogryczak 1999a; Ogryczak 2009) and their relations to robust solutions seems to be a promising research direction.

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