Sequential Algorithms for Exact and Approximate Max-Min Fair Bandwidth Allocation

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***Abstract*—Allocating bandwidth to maximize service flows with fair treatment of all the services is a key issue in network dimensioning. In such applications, the so-called Max-Min Fairness (MMF) solution concept is widely used. It is based on the worst service performance maximization with additional regularization by the lexicographic maximization of the second worst performance, the third one etc. The basic sequential procedure is applicable only for convex models, thus it allows to deal with basic design problems but fails if practical discrete restrictions commonly arriving in telecommunications network design are to be taken into account. We analyze alternative sequential approaches allowing to solve non-convex MMF network dimensioning problems. The directly defined sequential criteria can be introduced into the original model with some auxiliary variables and linear inequalities. The approaches guarantee the exact MMF solution for a complete set of criteria. However, they can be simplified by reducing the number of criteria thus generating effectively approximated MMF solutions.**

***Keywords*—Network design; network dimensioning; LP/ILP models; resource allocation; fairness; lexicographic optimization; lexicographic max-min**

I. Introduction

A fair way of the bandwidth distribution among competing demands becomes a key issue in computer networks [4] and telecommunications network design, in general [3], [19], [21], [8]. Due to increasing demand for Internet services, a problem of network dimensioning with elastic traffic arises which requires to allocate bandwidth to maximize service flows with fair treatment of all the services [19]. The problem of network dimensioning with elastic traffic can be formulated as follows [18]. Given a network topology $G =< V, E >$, consider a set of pairs of nodes as the set $J = \{1, 2, \ldots, m\}$ of services representing the elastic flow from source v^s_j to destination v^d_j. For each service, we have given the set P_j of possible routing paths in the network from the source to the destination represented in the form of binary matrices $\Delta_e = (\delta_{ejp})_{j\in J; p\in P_j}$ assigned to each link $e \in E$, where $\delta_{ejp} = 1$ if link e belongs to the routing path $p \in P_j$ (connecting v^s_j with v^d_j) and $\delta_{ejp} = 0$ otherwise. For each service $j \in J$, the elastic flow from source v^s_j to destination v^d_j is a variable representing the model outcome and it will be denoted by x_j. The flow may be realized along various paths $p \in P_j$ and modeled as $x_j = \sum_{p\in P_j} x_{jp}$ where x_{jp} are nonnegative variables representing the elastic flow from source v^s_j to destination v^d_j along the routing path $p \in P_j$. The single-path model requires additional multiple choice constraints to enforce nonbifurcated flows.

The network dimensioning problem depends on allocating the bandwidth to several links in order to maximize flows of all the services (demands). Typically, the network is already operated with some bandwidth preinstalled and decisions are rather related to the network expansion. Therefore, we assume that each link $e \in E$ has already capacity a_e while decision variable ξ_e represents the bandwidth newly allocated to link $e \in E$ thus expanding the link capacity to $a_e + \xi_e$. Certainly, all the decision variables must be nonnegative: $\xi_e \geq 0$ for all $e \in E$ and there are usually some bounds (upper limits) on possible expansion of the links capacities: $\xi_e \leq \bar{a}_e$ for all $e \in E$. Finally, the following constraints must be fulfilled:

$$0 \leq x_{jp} \leq M u_{jp},\ u_{jp} \in \{0,1\} \qquad j \in J;\ p \in P_j \tag{1a}$$

$$\sum_{p\in P_j} x_{jp} = x_j,\ \sum_{p\in P_j} u_{jp} = 1 \qquad j \in J \tag{1b}$$

$$\sum_{j\in J}\sum_{p\in P_j} \delta_{ejp} x_{jp} \leq a_e + \xi_e \qquad \forall e \in E \tag{1c}$$

$$0 \leq \xi_e \leq \bar{a}_e \qquad \forall e \in E \tag{1d}$$

$$\sum_{e\in E} c_e \xi_e \leq B \tag{1e}$$

where (1a)–(1b) define the total service flows representing single-path flow requirements using additional binary (flow assignment) variables u_{jp} and a sufficiently large constant M (upper bounding the largest possible flow x_j). Next, (1c) establishes the relation between service flows and links bandwidth. The quantity $x_e = \sum_{j\in J}\sum_{p\in P_j} \delta_{ejp} x_{jp}$ is the load of link e and it cannot exceed the available link capacity $a_e + \xi_e$. Further, while allocating the bandwidth to several links the decisions must keep the cost within available budget B (1e) where for each link $e \in E$ the unit cost of allocated bandwidth is c_e.

The network dimensioning model can be considered with various objectives depending on the chosen goal. Typically, the fairness requirement is formalized with the lexicographic maximinimization (lexicographic Max-Min approach). Within the telecommunications or network applications the lexicographic Max-Min approach has appeared already in [5] and

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now under the name Max-Min Fair (MMF) is treated as one of the standard fairness concepts [2], [7], [11], [19].

The lexicographic maximinimization can be seen as searching for a vector lexicographically maximal in the space of the feasible vectors with components rearranged in the non-decreasing order. This can be mathematically formalized as follows. Let $\langle \mathbf{x} \rangle = (x_{\langle 1 \rangle}, x_{\langle 2 \rangle}, \ldots, x_{\langle m \rangle})$ denote the vector obtained from $\mathbf{x}$ by rearranging its components in the non-decreasing order. That means $x_{\langle 1 \rangle} \leq x_{\langle 2 \rangle} \leq \ldots \leq x_{\langle m \rangle}$ and there exists a permutation π of set J such that $x_{\langle j \rangle} = x_{\pi(j)}$ for $j = 1, 2, \ldots, m$. Comparing lexicographically such ordered vectors $\langle \mathbf{x} \rangle$ one gets the so-called lex-min order. The MMF problem can be then represented in the following way:

$$\operatorname{lex\,max} \; \{(x_{\langle 1 \rangle}, x_{\langle 2 \rangle}, \ldots, x_{\langle m \rangle}) : \mathbf{x} \in A\} \qquad (2)$$

where A depicts the set of attainable outcomes defined with constraints (1). Actually, we focus our analysis on the MMF bandwidth allocation problem but the approaches developed can be applied to various lexicographic Max-Min optimization problems, i.e., to problem (2) with various attainable sets A.

The (point-wise) ordering of outcomes causes that the lexicographic Max-Min problem (2) is, in general, hard to implement. Note that the quantity $x_{\langle 1 \rangle}$ representing the worst outcome can be easily computed directly by the maximization $x_{\langle 1 \rangle} = \max \; \{r_1 : r_1 \leq x_j, \quad j \in J\}$. Similar simple formula does not exist for the further ordered outcomes $x_{\langle i \rangle}$. Nevertheless, for convex problems it is possible to build sequential algorithms for finding the consecutive values of the (unknown) MMF optimal outcome vector. While solving Max-Min problems for convex models there exists at least one blocked outcome which is constant on the entire set of optimal solutions to the Max-Min problem. Hence, the MMF solution can be found by solving a sequence of properly defined Max-Min problems with fixed outcomes (flows) that have been blocked by some critical constraints (link capacities) [10], [14]. Indeed, in the case of LP models it leads to efficient algorithms taking advantages of the duality theory for simple identification of blocked outcomes [1], [6], [20]. Unfortunately, in our network dimensioning model it applies only to the basic LP constraints (1b)–(1e). In the case of nonconvex feasible set, such a blocked quantity may not exist [12] which makes the approach not applicable to our case of nonbifurcated flows enforced by discrete constraints (1a)–(1b). This can be illustrated with the simplified network depicted in Fig. 1 with no capacity preinstalled $a_e = 0$ for all $e \in E$. The upper limits on possible expansion of the links capacities are given in the figure for each link. The cost coefficients are: 4 for link (v_1, v_3), 3 for (v_3, v_5) and all other equal 1, and the budget B=11. We consider two demands: one connecting v_1 with v_2 along two possible paths (v_1, v_2) or (v_1, v_3, v_4, v_2); second connecting v_5 with v_6 along two possible paths (v_5, v_6) or (v_5, v_3, v_4, v_6). The MMF solution is unique and it allocates flow 1 to path (v_1, v_2) (first demand) while flow 2 to path (v_3, v_5, v_6, v_4) (second demand). The Max-Min (single-path) problem leads us to the conclusion that one of two flows cannot exceed 1 but not allowing us to identify which one must be blocked. Note that the same difficulty arrives also for the single path problem without any budget constraint, though the optimal solution is then not unique.

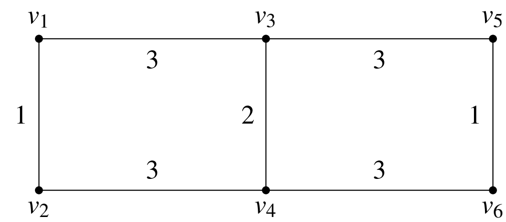

Fig. 1. Sample network without any critical link and blocked flow for Max-Min solution.

In this paper we analyze sequential approaches allowing to solve single-path (non-convex) MMF network dimensioning problems based on the lexicographic optimization of directly defined artificial criteria [16]. The criteria can be introduced into the original model with some auxiliary variables and linear inequalities independently from the problem structure. The approaches guarantee the exact MMF solution for a complete set of criteria and outperform the direct approach but still their applicability is limited to rather small networks [16]. Therefore, we focus our analysis on simplified approaches with reduced number of criteria thus generating effectively approximations to the MMF solutions.

II. Cumulated Ordered Outcomes

The point-wise ordering of outcomes for lexicographic optimization within the MMF problem (2) makes it hard for direct formulation. Nevertheless, with the use of auxiliary integer variables, any MMF problem (either convex or non-convex) can be formulated as the standard lexicographic maximization with Direct Ordered Outcomes (DOO) [22]

$$\begin{array}{ll} \operatorname{lex\,max} \; (r_1, r_2, \ldots, r_m) & \\ \text{s.t. } \mathbf{x} \in A & \\ r_i - x_j \leq C z_{ij}, \; z_{ij} \in \{0,1\} & i, j \in J \\ \sum_{j \in J} z_{ij} \leq i - 1 & i \in J. \end{array} \qquad (3)$$

where C is a sufficiently large constant (larger than any possible difference between various individual outcomes x_j) which allows us to enforce inequality $r_i \leq x_j$ for $z_{ij} = 0$ while ignoring it for $z_{ij} = 1$. Note that for $i = 1$ all binary variables z_{1j} are forced to 0 thus reducing the optimization in this case to the standard LP model. However, for any other $i > 1$ all m binary variables z_{ij} are an important part of the model thus contributing to exponential complexity of the implementation.

There is, however, a way to reformulate the MMF problem (2) so that only linear variables are used [14], [16]. It is based on the use of cumulated criteria $\bar{\theta}_i(\mathbf{x}) = \sum_{k=1}^{i} x_{\langle k \rangle}$ expressing, respectively: the worst (smallest) outcome, the total of the two worst outcomes, the total of the three worst outcomes, etc. Within the lexicographic optimization a cumulation of criteria does not affect the optimal solution and the MMF problem (2) can be formulated as the standard lexicographic maximization with cumulated ordered outcomes:

$$\operatorname{lex\,max} \; \{(\bar{\theta}_1(\mathbf{x}), \bar{\theta}_2(\mathbf{x}), \ldots, \bar{\theta}_m(\mathbf{x})) : \mathbf{x} \in A\} \qquad (4)$$

Note that for any given vector $\mathbf{x} \in \mathbb{R}^m$, the cumulated ordered value $\bar{\theta}_i(\mathbf{x})$ can be found as the optimal value of the following:

$$\bar{\theta}_i(\mathbf{x}) = \min\{\sum_{j\in J} x_j u_{ij} : \sum_{j\in J} u_{ij} = k,\ 0 \le u_{ij} \le 1\ \forall j \in J\}. \quad (5)$$

The above problem is an LP for a given outcome vector $\mathbf{x}$ while it becomes nonlinear for $\mathbf{x}$ being a variable. This difficulty is overcome by taking advantage of the LP duality [14], [16]. Indeed, the LP dual of problem (5) with variable r_i corresponding to the equation $\sum_{j\in J} u_{ij} = k$ and variables d_{ij} corresponding to upper bounds on u_{ij} leads us to the following formula:

$$\bar{\theta}_i(\mathbf{x}) = \max\{ir_i - \sum_{j\in J} d_{ij} : r_i - x_j \le d_{ij},\ d_{ij} \ge 0\ \ j \in J\}. \quad (6)$$

Thus $\bar{\theta}_i(\mathbf{x}) = \max\ \{ir_i - \sum_{j\in J}(x_j - r_i)_+ : \mathbf{x} \in A\ \}$ where $(.)_+$ denotes the nonnegative part of a number and r_i is an auxiliary (unbounded) variable.

Following (6), we may express the MMF problem (2) as a standard lexicographic optimization problem with predefined linear criteria:

$$\begin{aligned}&\operatorname{lex}\max(r_1 - \sum_{j\in J} d_{1j}, 2r_2 - \sum_{j\in J} d_{2j}, \ldots, mr_m - \sum_{j\in J} d_{mj})\\ &\text{s.t. } \mathbf{x} \in A,\ d_{ij} \ge r_i - x_j,\ d_{ij} \ge 0 \quad \forall i,j \in J.\end{aligned} \quad (7)$$

The above model is referred to as the Cumulated Ordered Outcomes (COO) approach [16]. An attainable outcome vector $\mathbf{x} \in A$ is an optimal solution of the MMF problem (2), if and only if it is an optimal solution of the COO model (7). Note that the direct lexicographic formulation of model COO remains valid for nonconvex (e.g. discrete) models, where the standard sequential approaches [9] are not applicable.

Although defined with simple linear constraints, model COO introduces $m^2 + m$ auxiliary variables and m^2 constraints. Thus, for many problems with not too large number of services (demands) m, problem (7) can easily be solved directly. However, it may cause a serious computational burden for real-life network dimensioning problems. The number of services (traffic demands), essentially, corresponds to the number of ordered pairs of network nodes which is already on the order of the square of the number of nodes $|V|$. Thus, finally the expanded multi-criteria model introduces $|V|^4$ variables and constraints which means polynomial but fast growth and may not be acceptable for larger networks. For instance, a rather small backbone network of a Polish ISP [17] consists of 12 nodes which leads to 132 possible services ($m = 132$) resulting in 132 criteria (lexicographic levels) and up to 17 424 deviational variables d_{ik} with corresponding constraints in last level problem. Actually, our earlier computational experiments with a simpler bandwidth allocation problem [16] has confirmed higher efficiency of the COO approach in comparison to the DOO model (3). Nevertheless, its applicability turned out to be limited to rather small networks with up to 20 services (elastic flows).

In order to reduce the problem size one may attempt to restrict the number of lexicographic levels in problem (7). Let us consider a sequence of indices $I = \{i_1, i_2, \ldots, i_q\} \subset J$, where $1 = i_1 < i_2 < \ldots < i_{q-1} < i_q = m$, and the corresponding restricted form of the lexicographic problem (4):

$$\operatorname{lex}\max\ \{(\bar{\theta}_{i_1}(\mathbf{x}), \bar{\theta}_{i_2}(\mathbf{x}), \ldots, \bar{\theta}_{i_q}(\mathbf{x})) : \mathbf{x} \in A\} \quad (8)$$

with only $q < m$ criteria. Following [16], full lexicographic model (4) allows us to generate the MMF solution. Reducing the number of criteria we restrict these capabilities. Nevertheless, one may still expect reasonably fair efficient solution and only *unfairness* may be related to the distribution of flows within classes of skipped criteria. In other words we have guaranteed some rough fairness approximating the MMF solution.

III. Multiple Level Throughputs

For some specific classes of discrete, or rather combinatorial, optimization problems, one may take advantage of the finiteness of the set of all possible outcome values. The ordered outcome vectors may be treated as describing a distribution of outcomes $\mathbf{x}$. In the case when there exists a finite set of all possible outcomes, we can directly describe the distribution of outcomes with frequencies of outcomes. Let $V = \{v_1, v_2, \ldots, v_r\}$ (where $v_1 < v_2 < \cdots < v_r$) denote the set of all attainable outcomes. We introduce integer functions $h_k(\mathbf{x})$ $(k = 1, \ldots, r)$ expressing the number of values v_k in the outcome vector $\mathbf{x}$. Having defined functions h_k we can introduce cumulative distribution functions:

$$\bar{h}_k(\mathbf{x}) = \sum_{l=1}^{k} h_l(\mathbf{x}) \quad k = 1, \ldots, r. \quad (9)$$

Function $\bar{h}_k$ expresses the number of outcomes smaller or equal to v_k. Since we want to maximize all the outcomes, we are interested in minimization of all functions $\bar{h}_k$. Indeed, the following assertion is valid [12]. For outcome vectors $\mathbf{x}', \mathbf{x}'' \in V^m$, $\langle \mathbf{x}' \rangle \ge \langle \mathbf{x}'' \rangle$ if and only if $\bar{h}_k(\mathbf{x}') \le \bar{h}_k(\mathbf{x}'')$ for all $k = 1, \ldots, r$. This equivalence allows to express the MMF problem (2) in terms of the standard lexicographic minimization problem with objectives $\bar{\mathbf{h}}(\mathbf{x})$ [14]:

$$\operatorname{lex}\min\ \{(\bar{h}_1(\mathbf{x}), \ldots, \bar{h}_r(\mathbf{x}))\ :\ \mathbf{x} \in A\}. \quad (10)$$

An attainable outcome vector $\mathbf{x} \in A$ is an optimal solution of the MMF problem (2), if and only if it is an optimal solution of the lexicographic problem (10).

Note that $\bar{h}_r(\mathbf{x}) = m$ for any $\mathbf{x}$ which means that the r-th criterion is always constant and therefore redundant in (10). Hence, the lexicographic problem (10) can be formulated as the following mixed integer problem:

$$\begin{aligned}&\operatorname{lex}\min\left(\sum_{j\in J} z_{1j},\ \sum_{j\in J} z_{2j}, \ldots,\ \sum_{j\in J} z_{r-1,j}\right)\\ &\text{s.t.}\\ &\mathbf{x} \in A,\ v_{k+1} - x_j \le C z_{kj},\ z_{kj} \in \{0,1\} \quad j \in J,\ k < r\end{aligned} \quad (11)$$

where C is a sufficiently large constant.

Taking advantage of possible weighting and cumulating achievements in lexicographic optimization, one may eliminate auxiliary integer variables from the objective functions. For this purpose we weight and cumulate vector $\bar{\mathbf{h}}(\mathbf{x})$ to get $\hat{h}_1(\mathbf{x}) = 0$ and:

$$\hat{h}_k(\mathbf{x}) = \sum_{l=1}^{k-1} (v_{l+1} - v_l)\bar{h}_l(\mathbf{x}) \quad k = 2,\dots,r. \tag{12}$$

Due to positive differences $v_{l+1} - v_l > 0$, the lexicographic minimization problem (10) is equivalent to the lexicographic problem with objectives $\hat{\mathbf{h}}(\mathbf{x})$:

$$\text{lex}\min\ \{(\hat{h}_1(\mathbf{x}),\dots,\hat{h}_r(\mathbf{x}))\ :\ \mathbf{x} \in A\} \tag{13}$$

Actually, vector function $\hat{\mathbf{h}}(\mathbf{x})$ provides a unique description of the distribution of coefficients of vector $\mathbf{x}$, i.e., for any $\mathbf{x}', \mathbf{x}'' \in V^m$ one gets: $\hat{\mathbf{h}}(\mathbf{x}') = \hat{\mathbf{h}}(\mathbf{x}'') \quad \Leftrightarrow \quad \langle \mathbf{x}' \rangle = \langle \mathbf{x}'' \rangle$. Moreover, $\hat{\mathbf{h}}(\mathbf{x}') \leq \hat{\mathbf{h}}(\mathbf{x}'')$ if and only if $\bar{\Theta}(\mathbf{x}') \geq \bar{\Theta}(\mathbf{x}'')$ [12]. Formula (12) allows one to express $\hat{h}_k(\mathbf{x})$ as a piecewise linear function $\hat{h}_k(\mathbf{x}) = \sum_{j\in J} \max\{v_k - x_j, 0\}$. This enables the formulation of the lexicographic minimization problem (13) as the so-called Shortfalls to Ordered Targets (SOT) model with auxiliary linear constraints [16].

Note that $\hat{h}_1(\mathbf{x}) = 0$ for any $\mathbf{x}$ which means that the first criterion is constant and redundant in problem (13). Moreover, $mv_r - \hat{h}_r(\mathbf{x}) = \sum_{j\in J} x_j$, thus representing the total throughput. Similarly, one may define for all k the complementary quantities $\eta_k(\mathbf{x}) = mv_k - \hat{h}_k(\mathbf{x}) = \sum_{j\in J} \min\{x_j, v_k\}$ expressing the corresponding partial throughputs generated by flows ranged to levels v_k [13]. Hence, the lexicographic minimization problem (13) is equivalent to the lexicographic problem with objectives $\eta(\mathbf{x})$:

$$\text{lex}\max\ \{(\eta_2(\mathbf{x}), \eta_3(\mathbf{x}),\dots,\eta_r(\mathbf{x}))\ :\ \mathbf{x} \in A\} \tag{14}$$

which leads us to the following assertion.

Theorem 1: An attainable outcome vector $\mathbf{x} \in A$ is an optimal solution of the MMF problem (2), if and only if it is an optimal solution of the lexicographic problem (14).

We will refer to model (14) as the Multiple Level Throughputs (MLT) approach. Note that the quantity $\eta_k(\mathbf{x})$ can be computed directly by the following minimization:

$$\eta_k(\mathbf{x}) = \max\{\sum_{j\in J} t_{kj} : t_{kj} \leq x_j,\ t_{kj} \leq v_k\ \ j \in J\}. \tag{15}$$

Hence, the entire MLT model (14) can be formulated as follows:

$$\begin{aligned} &\text{lex}\max \left(\sum_{j\in J} t_{2j},\ \sum_{j\in J} t_{3j},\dots,\ \sum_{j\in J} t_{rj} \right) \\ &\text{s.t.} \\ &\mathbf{x} \in A,\ t_{kj} \leq x_j,\ t_{kj} \leq v_k \quad j \in J, \quad k = 2,\dots,r. \end{aligned} \tag{16}$$

Note that the above formulation, unlike the problem (11), does not use integer variables and can be considered as an LP expansion of the original constraints (1).

The size of problem (16) depends on the number of different outcome values. Thus, for many problems with not too large number of outcome values, the problem can easily be solved directly. In many problems of telecommunications network design, the objective functions express the quality of service and one can easily consider a limited finite scale (grid) of the corresponding outcome values. However, in many cases the complete grid of flow values related to granulated capacity could extremely large. On the other hand, model (16) opens a way for the fuzzy representation of quality measures within the MMF problems.

Although defined with simple linear constraints, the expanded model (16) introduces $r \times m$ additional variables and inequalities. This may cause a serious computational burden for real-life network dimensioning problems. Note that the number of services (traffic demands) m corresponds to the number of ordered pairs of network nodes. On the other hand, quantity r represents the number of various possible outcomes (flow sizes). In order to reduce the problem size one may attempt to restrict the number of distinguished target values. Let us consider a sequence of indices $K = \{k_1, k_2, \dots, k_q\}$, where $v_{k_1} < v_{k_2} < \dots < v_{k_{q-1}} < v_{k_q}$, and the corresponding restricted form of the lexicographic model (14):

$$\text{lex}\max\ \{(\eta_{k_1}(\mathbf{x}),\dots,\eta_{k_q}(\mathbf{x}))\ :\ \mathbf{x} \in A\} \tag{17}$$

with only $q < r$ criteria. Following Theorem 1, model (14) allows us to generate the MMF solution. Reducing the number of criteria we restrict these capabilities. Nevertheless, we can essentially still expect reasonably fair solution and only *unfairness* may be related to the distribution of flows within classes of skipped criteria. In other words, we have guaranteed some rough fairness while it can be possibly improved by redistribution of flows within the intervals $(v_{k_j}, v_{k_{j+1}}]$ for $j = 1,2,\dots,q-1$.

IV. Computational experiments

We have analyzed the performance of approximate sequential approaches based on the Cumulated Ordered Outcomes (COO) model (8) and the Multiple Level Throughputs (MLT) model (16) with the condition $\mathbf{x} \in A$ representing the bandwidth allocation problem defined with constraints (1). For the COO model, besides exact MMF approach (7), we have also tested approximate MMF procedure in which only odd numbers of worst outcomes plus the number of all outcomes are sequentially optimized (i.e. $1,3,5,\dots,|J|$ worst outcomes). Such approach can be useful in case of larger problems. It is denoted further by COO2. We have not assumed any bandwidth granulation and thereby the grid of possible bandwidth values that can be allocated. Therefore, in case of the Multiple Level Throughputs approach the resulting bandwidth allocation is always an approximation to the exact MMF solution. However, to compare its computational effectiveness to that of COO and COO2, we use for MLT the same number of MMF steps (and distinct v_k values) as in COO or COO2, accordingly. The MLT approach with the reduced number of v_k values (similar to COO2) is denoted by MLT2. The v_k are computed with the formula $v_k = \underline{z} + \frac{(k-1)}{(|J|-1)}(\bar{z} - \underline{z})$, where $k = 1,2,3,\dots,|J|$ for MLT, or $k = 1,3,5,\dots,|J|$ for MLT2, $\underline{z}$ is the worst outcome (precomputed with max min) and $\bar{z}$ is a simple upper bound computed as $\max_{j\in J} \sum_{p\in P_j} \min_{e\in p}(a_e + \bar{a}_e)$.

TABLE I
COO (EXACT MMF): SOLUTION TIMES (SECONDS)

network	# of nodes	# of links	number of services 10	20	30	40	50
pdh	11	34	0.2	1.1	16.0	^{2}76.8	^{5}206.3
newyork	16	49	0.1	0.6	6.3	^{2}92.7	^{4}179.8
ta1	24	55	0.1	0.9	19.6	^{1}52.3	^{1}109.4
france	25	45	0.1	0.8	7.7	22.8	^{4}156.5
norway	27	51	0.1	1.2	4.8	41.9	^{3}148.8
cost266	37	57	0.0	0.6	3.4	18.8	53.3

TABLE II
COO2 (APPROXIMATE MMF): SOLUTION TIMES (SECONDS)

network	# of nodes	# of links	number of services 10	20	30	40	50
pdh	11	34	0.1	0.4	2.8	25.3	88.5
newyork	16	49	0.1	0.3	1.5	22.4	^{1}90.8
ta1	24	55	0.1	0.3	4.6	9.9	^{1}46.0
france	25	45	0.0	0.3	1.8	4.8	^{1}77.1
norway	27	51	0.0	0.5	1.6	8.6	^{1}72.7
cost266	37	57	0.1	0.2	1.0	3.7	8.4

TABLE III
MLT (APPROXIMATE MMF): SOLUTION TIMES (SECONDS)

network	# of nodes	# of links	number of services 10	20	30	40	50
pdh	11	34	0.1	0.8	3.5	16.9	41.5
newyork	16	49	0.1	0.7	3.4	13.7	48.5
ta1	24	55	0.1	0.5	4.6	10.2	15.3
france	25	45	0.1	0.5	2.7	5.4	15.4
norway	27	51	0.1	0.9	2.5	14.9	17.5
cost266	37	57	0.1	0.5	2.1	5.5	11.7

TABLE IV
MLT2 (APPROXIMATE MMF): SOLUTION TIMES (SECONDS)

network	# of nodes	# of links	number of services 10	20	30	40	50
pdh	11	34	0.1	0.3	0.8	6.5	6.8
newyork	16	49	0.0	0.3	1.0	4.8	7.5
ta1	24	55	0.0	0.2	1.1	3.1	21.6
france	25	45	0.0	0.1	1.0	2.0	6.0
norway	27	51	0.0	0.2	0.8	3.9	6.5
cost266	37	57	0.1	0.2	0.6	1.9	3.0

Each model has been computed using the standard sequential algorithm for lexicographic optimization with predefined objective functions. For lexicographic maximization problem $\operatorname{lex}\max\{(f_1(\mathbf{x}),\ldots,f_m(\mathbf{x})) : \mathbf{x} \in A\}$ the algorithm reads as follows:

Step 0: Put $k := 1$.

Step 1: Solve problem P_k:
$\max_{\mathbf{x}\in A}\{\tau_k : \tau_k \le f_k(\mathbf{x}), \tau_j^0 \le f_j(\mathbf{x}) \quad j < k\}$
denote the optimal solution by $(\mathbf{x}^0, \tau_k^0)$.

Step 2: If $k = m$, STOP ($\mathbf{x}^0$ is MMF optimal).
Otherwise, put $k := k+1$ and go to Step 1.

For the experiments we used 6 network topologies from the SNDLib (Survivable Network Design Library). For each topology 10 random problems were generated, as follows (all random numbers were generated with uniform distribution). First, for each link, its current capacity a_e and unit expansion cost c_e were generated as numbers in the range of 2 to 10 and 1 to 1.5, accordingly. Based on current capacity, the maximum expansion capacity $\bar{a}$ was generated as a number in the range of $0.2a_e$ to $0.6a_e$. The budget B for the network expansion (1e) was set to 130% of the current network value, i.e. $B = 1.3\sum_{e\in E}(c_e a_e)$. Then, random node pairs defining services were generated. For each service 3 different possible paths were chosen. Two of them were fully random and one was the shortest path between the end nodes (with smallest number of links).

All the tests have been performed on the Intel Core 2 Duo 2.4GHz computer employing the CPLEX 12.1 package, configured for using only one thread. Tables I–IV present solution times for the analyzed approaches. The times are averages of 10 randomly generated problems. The upper index (when pointed) denotes the number of tests for which the timeout of 300 seconds occurred. One can notice that the network structure influences performance but not in a consistent way – the solution times are not linearly dependent on the number of links. The solution times are much more affected by the number of services, and for larger problems only the MLT approaches give acceptable results in the sense of solving majority of problems within 300 seconds time limit (actually below one minute). Very promising are the results for the approximated approach with halved number of MMF steps.

TABLE V
APPROXIMATION ERROR FOR 10 SERVICES [%]

network	COO2 Q_3	COO2 Q_a	MLT Q_3	MLT Q_a	MLT2 Q_3	MLT2 Q_a
pdh	0.0	0.0	−2.1	1.6	−2.2	2.4
newyork	0.0	1.0	−2.8	2.6	−4.5	2.9
ta1	0.0	0.5	−2.0	1.4	−5.6	2.5
france	0.0	0.0	−4.9	0.6	−4.9	1.3
norway	0.0	0.1	−5.6	0.4	−6.2	0.4
cost266	0.0	1.5	−2.4	3.9	−2.8	3.9

To show what approximation errors are to be expected, for each test problem instance we computed the relative deviation from the exact solution (COO). Two solution parameters were considered: the sum (average) of 3 worst outcomes (denoted by Q_3) and the sum of all outcomes (total throughput denoted by Q_a). Only problem instances completely solved within the timeout limit (by all COO, COO2, MLT and MLT2)

TABLE VI
APPROXIMATION ERROR FOR 30 SERVICES [%]

network	COO2 Q_3	COO2 Q_a	MLT Q_3	MLT Q_a	MLT2 Q_3	MLT2 Q_a
pdh	0.0	0.3	−2.9	0.8	−2.9	2.3
newyork	0.0	0.1	−1.4	1.8	−1.5	2.1
ta1	0.0	0.1	−0.3	1.2	−0.3	1.8
france	0.0	0.1	−0.5	3.1	−0.5	6.2
norway	0.0	0.1	−0.8	2.6	−0.8	5.7
cost266	0.0	0.0	0.0	3.1	0.0	5.6

TABLE VII
APPROXIMATION ERROR FOR 50 SERVICES [%]

network	COO2 Q_3	COO2 Q_a	MLT Q_3	MLT Q_a	MLT2 Q_3	MLT2 Q_a
pdh	0.0	0.0	0.0	1.4	0.0	2.6
newyork	0.0	0.1	0.0	1.8	0.0	2.1
ta1	0.0	0.0	0.0	1.0	0.0	2.2
france	0.0	0.1	0.0	1.8	0.0	3.7
norway	0.0	0.1	0.0	3.6	0.0	6.8
cost266	0.0	0.0	0.0	3.0	0.0	6.2

were included into the average error computation. Thus the number of test instances can differ for each problem. The results, shown separately for different number of services, are presented in Tables V–VII. The most interesting is the very small approximation error for the COO2 approach, which suggests its usage in many practical applications. One should also notice the small error for the sum of 3 worst outcomes, especially for large problem instances.

V. CONCLUSION

The Max-Min Fair optimization can be implemented as the lexicographic optimization of directly defined artificial criteria introduced with some auxiliary variables and linear inequalities independently from the problem structure [14]. The approaches guarantee the exact MMF solution for a complete set of criteria and their applicability is limited to rather small networks [16]. We have developed some simplified sequential approaches with reduced number of criteria thus generating effectively approximations to the MMF solutions. Our computational analysis on the MMF single-path network dimensioning problems has shown that while the exact model COO can solve effectively problems with small number of services, the approximated Multiple Level Throughputs model enables to solve within a minute problems for networks with 30 nodes and 50 links providing very small approximation errors. It suggests its possible usage in many practical applications. Such performance is enough for efficient analysis of a country backbone network of ISP (12 nodes and 18 links in the case of Poland [17]). Nevertheless, further research is necessary on the models and corresponding algorithms tailored to specific MMF network optimization problems.

The models may also be applied to various MMF resource allocation problems, not necessarily related to networks.

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