On Decision Support Under Risk by the WOWA Optimization*

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Abstract. The problem of averaging outcomes under several scenarios to form overall objective functions is of considerable importance in decision support under uncertainty. The fuzzy operator defined as the so-called Weighted OWA (WOWA) aggregation offers a well-suited approach to this problem. The WOWA aggregation, similar to the classical ordered weighted averaging (OWA), uses the preferential weights assigned to the ordered values (i.e. to the worst value, the second worst and so on) rather than to the specific criteria. This allows one to model various preferences with respect to the risk. Simultaneously, importance weighting of scenarios can be introduced. In this paper we analyze solution procedures for optimization problems with the WOWA objective function. A linear programming formulation is introduced for optimization of the WOWA objective with monotonic preferential weights. Its computational efficiency is analyzed.

1 Introduction

Consider a decision problem under uncertainty where the decision is based on the maximization of a scalar (real valued) outcome. The final outcome is uncertain and only its realizations under various scenarios are known. Exactly, for each scenario S_i $(i = 1, \dots, m)$ the corresponding outcome realization is given as a function of the decision variables $y_i = f_i(\mathbf{x})$. We are interested in larger outcomes under each scenario. Hence, the decision under uncertainty can be considered a multiple criteria optimization problem:

$$\max \{ \ (f_1(\mathbf{x}), f_2(\mathbf{x}), \dots, f_m(\mathbf{x})) \ : \ \mathbf{x} \in \mathcal{F} \ \} \tag{1}$$

where $\mathbf{x}$ denotes a vector of decision variables to be selected within the feasible set $\mathcal{F} \subset R^q$, of constraints under consideration and $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \dots, f_m(\mathbf{x}))$ is a vector function that maps the feasible set $\mathcal{F}$ into the criterion space R^m.

* The research was supported by the Ministry of Science and Information Society Technologies under grant 3T11C 005 27 “Models and Algorithms for Efficient and Fair Resource Allocation in Complex Systems.”

K. Mellouli (Ed.): ECSQARU 2007, LNAI 4724, pp. 779–790, 2007.

From the perspective of decisions under uncertainty, model (1) only specifies that we are interested in maximization of all objective functions f_i for $i \in I = \{1, 2, \ldots, m\}$. In order to make it operational, one needs to assume some solution concept specifying what it means to maximize multiple objective functions. The solution concepts are defined by aggregation functions $a : R^m \rightarrow R$. Thus the multiple criteria problem (1) is replaced with the (scalar) maximization problem

$$\max \{a(\mathbf{f}(\mathbf{x})) \; : \; \mathbf{x} \in \mathcal{F}\}$$

The most commonly used aggregation is based on the weighted mean where positive importance weights p_i $(i = 1, \ldots, m)$ are allocated to several scenarios

$$A_{\mathbf{p}}(\mathbf{y}) = \sum_{i=1}^{m} y_i p_i \tag{2}$$

The weights are typically normalized to the total 1 ($\sum_{i=1}^{m} p_i = 1$) with possible interpretation as scenarios (subjective) probabilities. The weighted mean allowing to define the importance of scenarios does not allow one to model the decision maker's preferences regarding distribution of outcomes. The latter is crucial when aggregating various realizations of the same (uncertain) outcome under several scenarios one needs to model risk averse preferences [7].

The preference weights can be effectively introduced within the fuzzy optimization methodology with the so-called Ordered Weighted Averaging (OWA) aggregation developed by Yager [15]. In the OWA aggregation the weights are assigned to the ordered values (i.e. to the smallest value, the second smallest and so on) rather than to the specific criteria. This guarantees a possibility to model various preferences with respect to the risk. Since its introduction, the OWA aggregation has been successfully applied to many fields of decision making [18,19,6]. The weighting of the ordered outcome values causes that the OWA optimization problem is nonlinear even for linear programming (LP) formulation of the original constraints and criteria. Yager [16] has shown that the OWA optimization can be converted into a mixed integer programming problem. We have shown [10] that the OWA optimization with monotonic weights can be formed as a standard linear program of higher dimension.

The OWA operator allows one to model various aggregation functions from the maximum through the arithmetic mean to the minimum. Thus, it enables modeling of various preferences from the optimistic to the pessimistic one. On the other hand, the OWA does not allow one to allocate any importance weights to specific scenarios. Actually, the weighted mean (2) cannot be expressed in terms of the OWA aggregations. Torra [12] has incorporated importance weighting into the OWA operator within the Weighted OWA (WOWA) aggregation introduced as a particular case of Choquet integral using a distorted probability as the measure. The WOWA average becomes the weighted mean in the case of equal all the preference weights and it is reduced to the standard OWA average for equal all the importance weights. Since its introduction, the WOWA operator has been successfully applied to many fields of decision making [14] including metadata aggregation problems [1].

In this paper we analyze solution procedures for optimization problems with the WOWA objective functions. A linear programming formulation is introduced for optimization of the WOWA objective with increasing preferential weights thus representing risk averse preferences. The paper is organized as follows. In the next section we introduce formally the WOWA operator and derive some alternative computational formula based on direct application of the preferential weights to the conditional means according to the importance weights. In Section 3 we introduce the LP formulations for maximization of the WOWA aggregation with increasing weights. Finally, in Section 4 we demonstrate computational efficiency of the introduced models.

2 The Weighted OWA Aggregation

Let $\mathbf{w} = (w_1, \dots, w_m)$ be a weighting vector of dimension m such that $w_i \geq 0$ for $i = 1, \dots, m$ and $\sum_{i=1}^{m} w_i = 1$. The corresponding OWA aggregation of outcomes $\mathbf{y} = (y_1, \dots, y_m)$ can be mathematically formalized as follows [15]. First, we introduce the ordering map $\Theta : R^m \rightarrow R^m$ such that $\Theta(\mathbf{y}) = (\theta_1(\mathbf{y}), \theta_2(\mathbf{y}), \dots, \theta_m(\mathbf{y}))$, where $\theta_1(\mathbf{y}) \geq \theta_2(\mathbf{y}) \geq \cdots \geq \theta_m(\mathbf{y})$ and there exists a permutation τ of set I such that $\theta_i(\mathbf{y}) = y_{\tau(i)}$ for $i = 1, \dots, m$. Further, we apply the weighted sum aggregation to ordered achievement vectors $\Theta(\mathbf{y})$, i.e. the OWA aggregation has the following form:

$$A_{\mathbf{w}}(\mathbf{y}) = \sum_{i=1}^{m} w_i \theta_i(\mathbf{y}) \tag{3}$$

Yager [15] introduced a well appealing concept of the orness measure to characterize the OWA operators. orness($\mathbf{w}$) $= \sum_{i=1}^{m} \frac{m-i}{m-1} w_i$. For the max aggregation representing the fuzzy 'or' operator with weights $\mathbf{w} = (1, 0, \dots, 0)$ one gets orness($\mathbf{w}$) $= 1$ while for the min aggregation representing the fuzzy 'and' operator with weights $\mathbf{w} = (0, \dots, 0, 1)$ one has orness($\mathbf{w}$) $= 0$. For the average (arithmetic mean) one gets orness($(1/m, 1/m, \dots, 1/m)$) $= 1/2$. Actually, one may consider a complementary measure of andness defined as andness($\mathbf{w}$) $=$ $1 -$ orness($\mathbf{w}$). OWA aggregations with orness smaller or equal 1/2 are treated as and-like and they correspond to rather pessimistic (risk averse) preferences.

The OWA aggregations with increasing weights $w_1 \leq w_2 \leq \ldots \leq w_m$ define an and-like OWA operator. Actually, the andness properties of the OWA operators with increasing weights are total in the sense that they remain valid for any subaggregations defined by subsequences of their weights. Namely, for any $2 \leq k \leq m$ one gets $\sum_{j=1}^{k} \frac{k-j}{k-1} w_{i_j} \leq \frac{1}{2}$. Such total andness properties represent consequent risk averse preferences [7]. Therefore, we will refer to the OWA aggregation with increasing weights as the risk averse OWA.

Let $\mathbf{w} = (w_1, \dots, w_m)$ be an m-dimensional vector of preferential weights such that $w_i \geq 0$ for $i = 1, \dots, m$ and $\sum_{i=1}^{m} w_i = 1$. Further, let $\mathbf{p} = (p_1, \dots, p_m)$ be an m-dimensional vector of importance weights such that $p_i \geq 0$ for $i = 1, \dots, m$ and $\sum_{i=1}^{m} p_i = 1$. The corresponding Weighted OWA aggregation of outcomes $\mathbf{y} = (y_1, \dots, y_m)$ is defined [12] as follows:

$$A_{\mathbf{w},\mathbf{p}}(\mathbf{y}) = \sum_{i=1}^{m} \omega_i \theta_i(\mathbf{y}) \quad \text{with} \quad \omega_i = w^*(\sum_{k\le i} p_{\tau(k)}) - w^*(\sum_{k<i} p_{\tau(k)}) \qquad (4)$$

where w^* is an increasing function interpolating points $(\frac{i}{m}, \sum_{k\le i} w_k)$ together with the point (0.0) and τ representing the ordering permutation for $\mathbf{y}$ (i.e. $y_{\tau(i)} = \theta_i(\mathbf{y})$). Moreover, function w^* is required to be a straight line when the point can be interpolated in this way. We will focus our analysis on the piecewise linear interpolation function w^* which is the simplest form of the required interpolation.

The WOWA aggregation covers the standard weighted mean (2) with weights p_i as a special case of equal preference weights ($w_i = 1/m$ for $i = 1,\dots,m$). Actually, the WOWA operator is a particular case of Choquet integral using a distorted probability as the measure [3].

Example 1. Consider outcome vectors $\mathbf{y}' = (1,3,2,4,5)$ and $\mathbf{y}'' = (1,1,2,6,4)$ where individual outcomes correspond to five scenarios. While introducing preferential weights $\mathbf{w} = (0.05, 0.1, 0.15, 0.2, 0.5)$ one may calculate the OWA averages: $A_{\mathbf{w}}(\mathbf{y}') = 0.05\cdot 5 + 0.1\cdot 4 + 0.15\cdot 3 + 0.2\cdot 2 + 0.5\cdot 1 = 2$ and $A_{\mathbf{w}}(\mathbf{y}'') = 0.05\cdot 6 + 0.1\cdot 4 + 0.15\cdot 2 + 0.2\cdot 1 + 0.5\cdot 1 = 1.7$. Further, let us introduce importance weights $\mathbf{p} = (0.1, 0.1, 0.2, 0.5, 0.1)$ which means that results under the third scenario are 2 times more important then those under scenario 1, 2 or 5, while the results under scenario 4 are even 5 times more important. To take into account the importance weights in the WOWA aggregation (4) we introduce piecewise linear function

$$w^*(\xi) = \begin{cases} 0.05\xi/0.2 & \text{for } 0 \le \xi \le 0.2 \\ 0.05 + 0.10(\xi - 0.2)/0.2 & \text{for } 0.2 < \xi \le 0.4 \\ 0.15 + 0.15(\xi - 0.4)/0.2 & \text{for } 0.4 < \xi \le 0.6 \\ 0.3 + 0.2(\xi - 0.6)/0.2 & \text{for } 0.6 < \xi \le 0.8 \\ 0.5 + 0.5(\xi - 0.8)/0.2 & \text{for } 0.8 < \xi \le 1.0 \end{cases}$$

and calculate weights ω_i according to formula (4) as w^* increments corresponding to importance weights of the ordered outcomes, as illustrated in Fig. 1. In particular, one get $\omega_1 = w^*(p_5) = 0.025$ and $\omega_2 = w^*(p_5 + p_4) - w^*(p_5) = 0.275$ for vector $\mathbf{y}'$ while $\omega_1 = w^*(p_4) = 0.225$ and $\omega_2 = w^*(p_4 + p_5) - w^*(p_4) = 0.075$ for vector $\mathbf{y}''$. Finally, $A_{\mathbf{w},\mathbf{p}}(\mathbf{y}') = 0.025\cdot 5 + 0.275\cdot 4 + 0.1\cdot 3 + 0.35\cdot 2 + 0.25\cdot 1 = 2.475$ and $A_{\mathbf{w},\mathbf{p}}(\mathbf{y}'') = 0.225\cdot 6 + 0.075\cdot 4 + 0.2\cdot 2 + 0.25\cdot 1 + 0.25\cdot 1 = 2.55$.

Note that one may alternatively compute the WOWA values by using the importance weights to replicate corresponding scenarios and calculate then OWA aggregations. In the case of our importance weights $\mathbf{p}$ we need to consider five copies of scenario 4 and two copies of scenario 3 thus generating corresponding vectors $\tilde{\mathbf{y}}' = (1,3,2,2,4,4,4,4,4,5)$ and $\tilde{\mathbf{y}}'' = (1,1,2,2,6,6,6,6,6,4)$ of ten equally important outcomes. Original five preferential weights must be then applied respectively to the average of the two largest outcomes, the average of the next two largest outcomes etc. Indeed, we get $A_{\mathbf{w},\mathbf{p}}(\mathbf{y}') = 0.05\cdot 4.5 + 0.1\cdot 4 + 0.15\cdot 4 + 0.2\cdot 2.5 + 0.5\cdot 1.5 = 2.475$ and $A_{\mathbf{w},\mathbf{p}}(\mathbf{y}'') = 0.05\cdot 6 + 0.1\cdot 6 + 0.15\cdot 5 + 0.2\cdot 2 + 0.5\cdot 1 =$

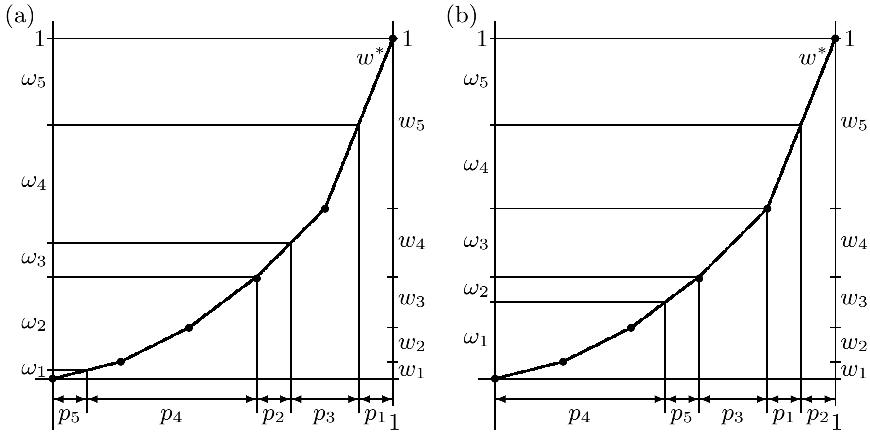

Fig. 1. Definition of weights ω_i for Example 1: (a) vector $\mathbf{y}'$, (b) vector $\mathbf{y}''$

2.55. We will further formalize this approach and take its advantages to build the LP computational models.

Function w^* can be defined by its generation function g with the formula $w^*(\alpha) = \int_0^\alpha g(\xi)\ d\xi$ Introducing breakpoints $\beta_i = \sum_{k\leq i} p_{\tau(k)}$ and $\beta_0 = 0$ allows us to express $\omega_i = \int_0^{\beta_i} g(\xi)\ d\xi - \int_0^{\beta_{i-1}} g(\xi)\ d\xi = \int_{\beta_{i-1}}^{\beta_i} g(\xi)\ d\xi$ and the entire WOWA aggregation as

$$A_{\mathbf{w},\mathbf{p}}(\mathbf{y}) = \sum_{i=1}^{m} \theta_i(\mathbf{y}) \int_{\beta_{i-1}}^{\beta_i} g(\xi)\ d\xi = \int_0^1 g(\xi) \overline{F}_{\mathbf{y}}^{(-1)}(\xi)\ d\xi \tag{5}$$

where $\overline{F}_{\mathbf{y}}^{(-1)}$ is the stepwise function $\overline{F}_{\mathbf{y}}^{(-1)}(\xi) = \theta_i(\mathbf{y})$ for $\beta_{i-1} < \xi \leq \beta_i$. It can also be mathematically formalized as follows. First, we introduce the right-continuous cumulative distribution function (cdf):

$$F_{\mathbf{y}}(d) = \sum_{i=1}^{m} p_i \delta_i(d) \quad \text{where} \quad \delta_i(d) = \begin{cases} 1 & \text{if } y_i \leq d \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

which for any real (outcome) value d provides the measure of outcomes smaller or equal to d. Next, we introduce the quantile function $F_{\mathbf{y}}^{(-1)} = \inf\ \{\eta : F_{\mathbf{y}}(\eta) \geq \xi\}$ for $0 < \xi \leq 1$ as the left-continuous inverse of the cumulative distribution function $F_{\mathbf{y}}$, and finally $\overline{F}_{\mathbf{y}}^{(-1)}(\xi) = F_{\mathbf{y}}^{(-1)}(1-\xi)$.

Formula (5) provides the most general expression of the WOWA aggregation allowing for expansion to continuous case. The original definition of WOWA allows one to build various interpolation functions w^* [13] thus to use different generation functions g in formula (5). We have focused our analysis on the the piecewise linear interpolation function w^*. Note, however, that the piecewise linear functions may be built with various number of breakpoints, not necessarily

m. Thus, any nonlinear function can be well approximated by an piecewise linear function with appropriate number of breakpoints. Therefore, we will consider weights vectors $\mathbf{w}$ of dimension n not necessarily equal to m. Any such piecewise linear interpolation function w^* can be expressed with the stepwise generation function $g(\xi) = nw_k$ for $(k-1)/n < \xi \leq k/n$, $k = 1, \dots, n$. This leads us to the following specification of formula (5):

$$A_{\mathbf{w},\mathbf{p}}(\mathbf{y}) = \sum_{k=1}^{n} w_k n \int_{(k-1)/n}^{k/n} \overline{F}_{\mathbf{y}}^{(-1)}(\xi)\, d\xi = \sum_{k=1}^{n} w_k n \int_{(k-1)/n}^{k/n} F_{\mathbf{y}}^{(-1)}(1-\xi)\, d\xi \quad (7)$$

Note that $n \int_{(k-1)/n}^{k/n} \overline{F}_{\mathbf{y}}^{(-1)}(\xi)\, d\xi$ represents the average within the k-th portion of $1/n$ largest outcomes, the corresponding conditional mean [9,11]. Hence, formula (7) defines WOWA aggregations with preferential weights $\mathbf{w}$ as the corresponding OWA aggregation but applied to the conditional means calculated according to the importance weights $\mathbf{p}$ instead of the original outcomes. Fig. 2 illustrates application of formula (7) for computation of the WOWA aggregations in Example 1.

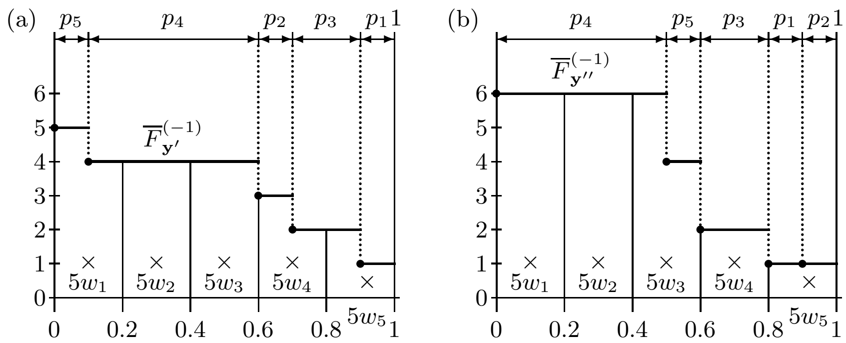

Fig. 2. Formula (7) applied to calculations in Example 1: (a) vector $\mathbf{y}'$, (b) vector $\mathbf{y}''$

We will treat formula (7) as a formal definition of the WOWA aggregation of m-dimensional outcomes $\mathbf{y}$ defined by m-dimensional importance weights $\mathbf{p}$ and n-dimensional preferential weights $\mathbf{w}$. We will focus our analysis on the WOWA aggregation defined by increasing weights $w_1 \leq w_2 \leq \ldots \leq w_n$. Following formula (7), maximization of such WOWA aggregation models risk averse preferences since equally important unit of a smaller outcome is considered with a larger weight. This is mathematically represented by the convexity of function w^* as well as it may be viewed as andness of the WOWA operator [4] when considered as the OWA defined via the regular increasing monotone (RIM) quantifiers [17] ($\int_0^1 w^*(\xi)d\xi \leq 0.5$).

3 The LP Model for WOWA Optimization

Formula (7) defines the WOWA value applying preferential weights w_i to importance weighted averages within quantile intervals. It may reformulated to use the tail averages

$$A_{\mathbf{w},\mathbf{p}}(\mathbf{y}) = \sum_{k=1}^{n} n w_k (L(\mathbf{y},\mathbf{p},1-\frac{k-1}{n}) - L(\mathbf{y},\mathbf{p},1-\frac{k}{n})) = \sum_{k=1}^{n} w'_k L(\mathbf{y},\mathbf{p},\frac{k}{n}) \quad (8)$$

where $L(\mathbf{y},\mathbf{p},\xi)$ is defined by left-tail integrating of $F_{\mathbf{y}}^{(-1)}$, i.e.

$$L(\mathbf{y},\mathbf{p},0) = 0 \quad \text{and} \quad L(\mathbf{y},\mathbf{p},\xi) = \int_0^{\xi} F_{\mathbf{y}}^{(-1)}(\alpha) d\alpha \quad \text{for } 0 < \xi \leq 1 \quad (9)$$

while weights $w'_k = n(w_{n-k+1} - w_{n-k})$ for $k = 1, \ldots, n-1$ and $w'_n = n w_1$.

Graphs of functions $L(\mathbf{y},\mathbf{p},\xi)$ (with respect to ξ) take the form of convex piecewise linear curves, the so-called absolute Lorenz curves [8] connected to the relation of the second order stochastic dominance (SSD). Therefore, formula (8) relates the WOWA average to the SSD consistent risk measures based on the tail means [5] provided that the importance weights are treated as scenario probabilities.

Following (8), maximization of a risk averse WOWA aggregation defined by increasing weights $w_1 \leq w_2 \leq \ldots \leq w_n$

$$\max\{A_{\mathbf{w},\mathbf{p}}(\mathbf{y}) \ : \ \mathbf{y} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \mathcal{F}\} \quad (10)$$

results in problem

$$\max\{\sum_{k=1}^{n} w'_k L(\mathbf{y},\mathbf{p},\frac{k}{n}) \ : \ \mathbf{y} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \mathcal{F}\}$$

with positive weights w'_k.

According to (9), values of function $L(\mathbf{y},\mathbf{p},\xi)$ for any $0 \leq \xi \leq 1$ can be given by optimization:

$$L(\mathbf{y},\mathbf{p},\xi) = \min_{s_i} \{\sum_{i=1}^{m} y_i s_i : \sum_{i=1}^{m} s_i = \xi, \quad 0 \leq s_i \leq p_i \quad \forall\ i\ \} \quad (11)$$

The above problem is an LP for a given outcome vector $\mathbf{y}$ while it becomes nonlinear for $\mathbf{y}$ being a vector of variables. This difficulty can be overcome by taking advantage of the LP dual to (11). Introducing dual variable t corresponding to the equation $\sum_{i=1}^{m} s_i = \xi$ and variables d_i corresponding to upper bounds on s_i one gets the following LP dual expression of $L(\mathbf{y},\mathbf{p},\xi)$

$$L(\mathbf{y},\mathbf{p},\xi) = \max_{t,d_i} \{\xi t - \sum_{i=1}^{m} p_i d_i : t - d_i \leq y_i, \ d_i \geq 0 \quad \forall\ i\} \quad (12)$$

Therefore, maximization of the WOWA aggregation (10) can be expressed as follows

$$\begin{aligned}
\max_{t_k, d_{ik}, y_i, x_j} \; & \sum_{k=1}^{n} w'_k [\frac{k}{n} t_k - \sum_{i=1}^{m} p_i d_{ik}] \\
\text{s.t. } & t_k - d_{ik} \leq y_i, \;\; d_{ik} \geq 0 \quad \forall\, i, k \\
& \mathbf{y} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \mathcal{F}
\end{aligned}$$

Consider multiple criteria problems (1) with linear objective functions $f_i(\mathbf{x}) = \mathbf{c}_i \mathbf{x}$ and polyhedral feasible sets:

$$\max\{(y_1, y_2, \ldots, y_m)) \;:\; \mathbf{y} = \mathbf{C}\mathbf{x}, \quad \mathbf{A}\mathbf{x} = \mathbf{b}, \quad \mathbf{x} \geqq \mathbf{0}\} \tag{13}$$

where $\mathbf{C}$ is an $m \times q$ matrix (consisting of rows $\mathbf{c}_i$), $\mathbf{A}$ is a given $r \times q$ matrix and $\mathbf{b} = (b_1, \ldots, b_r)^T$ is a given RHS vector. For such problems, we get the following LP formulation of the WOWA maximization (10):

$$\max_{t_k, d_{ik}, y_i, x_j} \; \sum_{k=1}^{n} \frac{k}{n} w'_k t_k - \sum_{k=1}^{n} \sum_{i=1}^{m} w'_k p_i d_{ik} \tag{14}$$

$$\text{s.t. } \; \mathbf{A}\mathbf{x} = \mathbf{b} \tag{15}$$

$$\mathbf{y} - \mathbf{C}\mathbf{x} = \mathbf{0} \tag{16}$$

$$d_{ik} \geq t_k - y_i \quad \forall\, i, k \tag{17}$$

$$d_{ik} \geq 0 \quad \forall\, i, k; \;\; x_j \geq 0 \;\forall\, j \tag{18}$$

Model (14)–(18) is an LP problem with $mn+m+n+q$ variables and $mn+m+r$ constraints. Thus, for problems with not too large number of scenarios (m) and preferential weights (n) it can be solved directly. Note that WOWA model (14)–(18) differs from the analogous deviational model for the OWA optimizations [10] only due to coefficients within the objective function (14) and the possibility of different values of m and n.

The number of constraints in problem (14)–(18) is similar to the number of variables. Nevertheless, for the simplex approach it may be better to deal with the dual of (14)–(18) than with the original problem. Note that variables d_{ik} in the primal are represented with singleton columns. Hence, the corresponding rows in the dual represent only simple upper bounds.

Introducing the dual variables: $\mathbf{u} = (u_l)_{l=1,\ldots,r}$, $\mathbf{v} = (v_i)_{i=1,\ldots,m}$ and $\mathbf{z} = (z_{ik})_{i=1,\ldots,m;\; k=1,\ldots,n}$ corresponding to the constraints (15), (16) and (17), respectively, we get the following dual:

$$\begin{aligned}
\min_{z_{ik}, v_i, u_l} \; & \mathbf{u}\mathbf{b} \\
\text{s.t. } \; & \mathbf{u}\mathbf{A} - \mathbf{v}\mathbf{C} \geqq \mathbf{0} \\
& v_i - \sum_{k=1}^{n} z_{ik} = 0 \quad \forall\, i \\
& \sum_{i=1}^{m} z_{ik} = \frac{k}{n} w'_k \quad \forall\, k \\
& 0 \leq z_{ik} \leq p_i w'_k \quad \forall\, i, k
\end{aligned} \tag{19}$$

The dual problem (19) contains: $m+n+q$ structural constraints, $r+m$ unbounded variables and mn bounded variables. Since the average complexity of the simplex method depends on the number of constraints, the dual model (19) can be directly solved for quite large values of m and n. Moreover, the columns corresponding to mn variables z_{ik} form the transportation/assignment matrix thus allowing one to employ special techniques of the simplex SON algorithm [2] for implicit handling of these variables. Such techniques increase dramatically efficiency of the simplex method but they require a special tailored implementation. We have not tested this approach within our initial computational experiments based on the use of a general purpose LP code.

4 Computational Tests

In order to analyze the computational performances of the LP model for the WOWA optimization, similarly to [10], we have solved randomly generated problems of portfolio optimization according to the (discrete) scenario analysis approach [6]. There is given a set of securities for an investment $J = \{1, 2, \dots, q\}$. We assume, as usual, that for each security $j \in J$ there is given a vector of data $(c_{ij})_{i=1,\dots,m}$, where c_{ij} is the observed (or forecasted) rate of return of security j under scenario i (hereafter referred to as outcome). We consider discrete distributions of returns defined by the finite set $I = \{1, 2, \dots, m\}$ of scenarios with the assumption that each scenario can be assigned the importance weight p_i that can be seen as the subjective probability of the scenario. The outcome data forms an $m \times q$ matrix $\mathbf{C} = (c_{ij})_{i=1,\dots,m;j=1,\dots,q}$ whose columns correspond to securities while rows $\mathbf{c}_i = (c_{ij})_{j=1,2,\dots,q}$ correspond to outcomes. Further, let $\mathbf{x} = (x_j)_{j=1,2,\dots,q}$ denote the vector of decision variables defining a portfolio. Each variable x_j expresses the portion of the capital invested in the corresponding security. Portfolio $\mathbf{x}$ generates outcomes

$$\mathbf{y} = \mathbf{C}\mathbf{x} = (\mathbf{c}_1\mathbf{x}, \mathbf{c}_2\mathbf{x}, \dots, \mathbf{c}_m\mathbf{x})$$

The portfolio selection problem can be considered as an LP problem with m uniform objective functions $f_i(\mathbf{x}) = \mathbf{c}_i\mathbf{x} = \sum_{j=1}^{q} c_{ij}x_j$ to be maximized [6]:

$$\max \{\mathbf{C}\mathbf{x} \; : \; \sum_{j=1}^{q} x_j = 1, \quad x_j \geq 0 \quad \text{for } j = 1, \dots, q\}$$

Hence, our portfolio optimization problem can be considered a special case of the multiple criteria problem and one may seek an optimal portfolio with some criteria aggregation. Note that the aggregation must take into account the importance of various scenarios thus allowing importance weights p_i to be assigned to several scenarios. Further the preferential weights w_k must be increasing to represent the risk averse preferences (more attention paid on improvement of smaller outcomes). Thus we get the WOWA maximization problem

$$\max \{A_{\mathbf{w},\mathbf{p}}(\mathbf{f}(\mathbf{x})) \; : \; \sum_{j=1}^{q} x_j = 1, \quad x_j \geq 0 \quad \text{for } j = 1, \dots, q\} \tag{20}$$

Our computational tests were based on the randomly generated problems (20) with varying number q of securities (decision variables) and number m of scenarios. The generation procedure worked as follows. First, for each security j the maximum rate of return r_j was generated as a random number uniformly distributed in the interval $[0.05, 0.15]$. Next, this value was used to generate specific outcomes c_{ij} (the rate of return under scenarios i) as random variables uniformly distributed in the interval $[-0.75r_j, r_j]$. Further, strictly increasing and positive weights w_k were generated. The weights were not normalized which allowed us to define them by the corresponding increments $\delta_k = w_k - w_{k-1}$. The latter were generated as uniformly distributed random values in the range of 1.0 to 2.0, except from a few (5 on average) possibly larger increments ranged from 1.0 to $n/3$. Importance weights p_i were generated according to the exponential smoothing scheme, which assigns exponentially decreasing weights to older or subjectively less probable scenarios: $p_i = \alpha(1-\alpha)^{i-1}$ for $i = 1, 2, \dots, m$ and the parameter α is chosen for each test problem size separately to keep the value of p_m around 0.001.

We tested solution times for different size parameters m and q. The basic tests were performed for the standard WOWA model with $n = m$. However, we also analyzed the case of larger n for more detailed preferences modeling, as well as the case of smaller n thus representing a rough preferences model. For each number of decision variables (securities) q and number of criteria (scenarios) m we solved 10 randomly generated problems (20). All computations were performed on a PC with the Pentium 1.7GHz processor employing the CPLEX 9.1 package. The 120 seconds time limit was used in all the computations.

Table 1. Solution times [s] for the primal model (14)–(18)

Number of	Number of variables (q)							
scenarios (m)	10	20	50	100	150	200	300	400
10	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
20	0.0	0.0	0.0	0.1	0.1	0.1	0.1	0.1
50	1.8	2.5	3.5	4.0	4.1	4.0	3.9	4.0
100	55.7	77.4	89.5	^{2}106.3	^{7}117.7	–	–	–

In Tables 1 and 2 we show the solution times for the primal (14)–(18) and the dual (19) forms of the computational model, being the averages of 10 randomly generated problems. Upper index in front of the time value indicates the number of tests among 10 that exceeded the time limit. The empty cell (minus sign) shows that this occurred for all 10 instances. Both forms were solved by the CPLEX code without taking advantages of the constraints structure specificity. The dual form of the model performs much better in each tested problem size. It behaves very well with increasing number of variables if the number of scenarios does not exceed 100. Similarly, the model performs very well with increasing number of scenarios if only the number of variables does not exceed 20.

Table 2. Solution times [s] for the dual model (19)

Number of scenarios (m)	Number of variables (q)							
	10	20	50	100	150	200	300	400
10	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
20	0.0	0.0	0.1	0.0	0.0	0.0	0.0	0.1
50	0.1	0.1	0.4	0.7	0.9	1.0	1.5	1.9
100	0.7	1.0	3.4	19.5	24.9	30.0	33.6	38.9
150	2.3	3.5	7.9	80.2	–	–	–	–
200	5.6	7.9	17.1	–	–	–	–	–
300	20.0	30.6	[1]89.3	–	–	–	–	–
400	51.9	92.8	–	–	–	–	–	–

Table 3. Solution times [s] for different numbers of preferential weights ($m = 100$, $q = 50$)

Number of preferential weights (n)								
3	5	10	20	50	100	150	200	300
0.0	0.1	0.1	0.4	3.3	3.4	2.6	3.6	6.5

Table 3 presents solution times for different numbers of the preferential weights for problems with 100 scenarios and 50 variables. It can be noticed that the computational efficiency can be improved by reducing the number of preferential weights which can be reasonable in non-automated decision making support systems and actually provides very good results for portfolio optimization problems [5]. On the other hand increasing the number of preferential weights and thus the number of breakpoints in the interpolation function does not induce the massive increase in the computational complexity.

5 Concluding Remarks

The problem of averaging outcomes under several scenarios to form overall objective functions is of considerable importance in decision support under uncertainty. The WOWA aggregation [12] represents such a universal tool allowing one to take into account both the preferential weights allocated to ordered outcomes and the importance weights allocated to several scenarios. The ordering operator used to define the WOWA aggregation is, in general, hard to implement. We have shown that the WOWA aggregations with the increasing weights can be modeled by introducing auxiliary linear constraints. Hence, an LP problem with the risk averse WOWA aggregation can be formed as a standard linear program and it can be further simplified by taking advantages of the LP duality.

Initial computational experiments show that the formulation enables to solve effectively medium size problems. Actually, the number of 100 scenarios covered by the dual approach to the LP model seems to be quite enough for most applications, including the fuzzy aggregations and decisions under risk. The problems have been solved directly by general purpose LP code. Taking advantages of the constraints structure specificity may remarkably extend the solution capabilities. In particular, the simplex SON algorithm [2] may be used for exploiting the LP embedded network structure in the dual form of the model.

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