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Decision Aiding

On solving linear programs with the ordered weighted averaging objective

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Abstract

The problem of aggregating multiple criteria to form overall objective functions is of considerable importance in many disciplines. The most commonly used aggregation is based on the weighted sum. The ordered weighted averaging (OWA) aggregation, introduced by Yager, uses the weights assigned to the ordered values (i.e. to the worst value, the second worst and so on) rather than to the specific criteria. This allows to model various aggregation preferences, preserving simultaneously the impartiality (neutrality) with respect to the individual criteria. In this paper we analyze solution procedures for linear programs with the OWA objective functions. Two alternative linear programming formulations are introduced and their computational efficiency is analyzed.

Keywords: Multiple criteria; Ordered weighted averaging; Lexicographic maximin; Linear programming; Equity

1. Introduction

Consider a decision problem defined as a linear optimization problem with m uniform objective functions $f_i(\mathbf{x}) = \mathbf{c}^i\mathbf{x}$. For simplification we assume, without loss of generality, that the objective functions are to be maximized. The problem can be formulated as follows:

$$\max\{\mathbf{Cx} : \mathbf{x} \in Q\} \tag{1}$$

where $\mathbf{C}$ is an $m \times n$ matrix (consisting of rows $\mathbf{c}^i$) representing the vector-function that maps the decision space $X = R^n$ into the criterion space $Y = R^m$, $\mathbf{x} \in X$ denotes the vector of decision variables, $Q \subset X$ denotes the feasible set defined by a system of linear equations with nonnegative variables

$$Q = \{\mathbf{x} \in R^n : \mathbf{Ax} = \mathbf{b}, \mathbf{x} \geqq \mathbf{0}\} \tag{2}$$

where $\mathbf{A}$ is a given $p \times n$ matrix and $\mathbf{b} = (b_1, \ldots, b_p)^{\mathrm{T}}$ is a given RHS vector.

We refer to the elements of the criterion space as achievement vectors. An achievement vector $\mathbf{y} \in Y$ is attainable if it expresses outcomes of a feasible solution $\mathbf{x} \in Q$ ($\mathbf{y} = \mathbf{Cx}$). Model (1) only specifies that we are interested in maximization of all individual objective functions f_i for $i \in I = \{1, 2, \ldots, m\}$. Each feasible solution for which one cannot improve any criterion without worsening another is called an *efficient* (Pareto-optimal) solution [14].

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In order to make model (1) operational, one needs to assume some solution concept specifying what it means to maximize multiple objective functions. The solution concepts are defined by aggregation functions $a : Y \rightarrow R$ to be maximized. Thus the multiple criteria problem (1) is replaced with the maximization problem

$$\max\{a(\mathbf{Cx}) : \mathbf{x} \in Q\} \tag{3}$$

In order to guarantee the consistency of the aggregated problem (3) with maximization of all individual objective functions in the original multiple criteria problem, the aggregation function must be *strictly monotonic*, i.e.

$$y'_i < y_i \Rightarrow a(y_1, \ldots, y_{i-1}, y'_i, y_{i+1}, \ldots, y_m) < a(y_1, y_2, \ldots, y_m) \quad \text{for } i \in I \tag{4}$$

In the case of a strictly monotonic function a, every optimal solution to the aggregated problem (3) is an efficient solution of the original multiple criteria problem.

The most commonly used aggregation is based on the weighted sum

$$a(\mathbf{y}) = \sum_{i=1}^{m} w_i y_i \tag{5}$$

In the case of positive weights ($w_i > 0$ for $i \in I$), every optimal solution to the weighted sum aggregation (i.e. problem (3) with the aggregation function (5)) is an efficient solution of the original multiple criteria problem. Moreover, for linear multiple criteria problems, we consider, every efficient solution can be found as an optimal solution to the weighted sum aggregation with appropriate positive weights (cf. [14]).

A primary factor in determination of the aggregation structure is the relationship between the criteria involved. There are several multiple criteria decision problems where the weighted sum aggregation cannot be applied since the criteria are uniform and the distribution of their values is an important issue. Such problems require an aggregation function satisfying the property of *impartiality* (neutrality, symmetry)

$$a(y_{\tau(1)}, y_{\tau(2)}, \ldots, y_{\tau(m)}) = a(y_1, y_2, \ldots, y_m) \quad \text{for any permutation } \tau \text{ of } I \tag{6}$$

The most direct problems with uniform impartial criteria are related to the optimization of systems which serve many users. For instance, in location problems the decisions often concern the placement of facilities in positions so that the clients are treated impartially [11]. Another type of model is that of approximation of discrete data set by a functional form. The residuals may be viewed as criteria to be minimized, and in the classical approaches, there is no reason to treat them in any way but impartially. The latter class of problems covers as special cases the problems arriving in neural networks and fuzzy optimization methodologies [21] as well as in the goal programming and the reference point approaches to the multiple criteria decision support [15]. Uniform objectives arise in many dynamic programs where individual objective functions represent the same outcome for various periods [8]. In the stochastic problems uniform objectives may represent various possible values of the same (uncertain) outcome under several scenarios.

The weighted sum aggregation (5) violates the requirement of impartiality as it assigns the weights to the specific criteria. Yager [16] introduced the so-called ordered weighted averaging (OWA) aggregation. In the OWA aggregation the weights are assigned to the ordered values (i.e. to the smallest value, the second smallest and so on) rather than to the specific criteria. This can be mathematically formalized as follows. We introduce the ordering map Θ: $R^m \rightarrow R^m$ such that $\Theta(\mathbf{y}) = (\theta_1(\mathbf{y}), \theta_2(\mathbf{y}), \ldots, \theta_m(\mathbf{y}))$, where $\theta_1(\mathbf{y}) \leqslant \theta_2(\mathbf{y}) \leqslant \cdots \leqslant \theta_m(\mathbf{y})$ and there exists a permutation τ of set I such that $\theta_i(\mathbf{y}) = y_{\tau(i)}$ for $i = 1, 2, \ldots, m$. Further, we apply the weighted sum aggregation to ordered achievement vectors $\Theta(\mathbf{y})$, i.e. the OWA aggregation function has the form:

$$a_{\mathbf{w}}(\mathbf{y}) = \sum_{i=1}^{m} w_i \theta_i(\mathbf{y}) \tag{7}$$

Note that formula (7) differs from that originally introduced by Yager [16], due to opposite ordering of outcomes (the weight w_1 corresponds to the largest outcome $\theta_m(\mathbf{y})$ in [16]) and not necessarily normalized weights ($\sum_{i=1}^{m} w_i = 1$ in [16]). These changes, without loss of generality, simplify the cumulative transformations we introduce further.

The OWA aggregation (7) allows to model various aggregation functions from the minimum ($w_1 = 1$, $w_i = 0$ for $i = 2, \dots, m$) trough the arithmetic mean ($w_i = 1/m$ for $i = 1, \dots, m$) to the maximum ($w_m = 1$, $w_i = 0$ for $i = 1, \dots, m-1$). Since its introduction, the OWA aggregation has been applied to many fields of decision making [3,12, 17,19,20]. In the case of positive weights ($w_i > 0$ for $i \in I$), the OWA aggregation (7) is strictly monotonic and impartial.

When applying the OWA aggregation to problem (1) we get

$$\max \left\{ \sum_{i=1}^{m} w_i \theta_i(\mathbf{Cx}) : \mathbf{x} \in Q \right\} \tag{8}$$

In this paper we analyze solution procedures for problem (8). The ordering operator Θ causes that the OWA optimization problem (8) is nonlinear even for the case of linear programming (LP) form of the original constraints and criteria. Yager [17] has shown that the nature of the nonlinearity introduced by the ordering operations allows us to convert the optimization (8) into a mixed integer programming problem. We show that the OWA optimization with the monotonic weights can be formed as a standard LP of higher dimension.

The paper is organized as follows. In the next section we review properties of the OWA aggregation with the monotonic weights. In Section 3 we introduce and discuss two alternative LP formulations of the OWA optimization. Their computational efficiency is further analyzed in Section 4.

2. Equitable OWA aggregations

The ordering operator Θ causes that the OWA optimization problem (8) is nonlinear and hard to implement. Note that the quantity $\theta_1(\mathbf{y})$ representing the worst outcome can be easily computed directly by the LP maximization:

$$\begin{aligned} &\theta_1(\mathbf{y}) = \max r_1 \\ &\text{subject to} \\ &\quad r_1 \leqslant y_i \quad \text{for} \quad i = 1, 2, \dots, m \end{aligned}$$

Following Yager [17], similar formula can be given for any $\theta_k(\mathbf{y})$ although requiring the use of integer variables. Namely, for any $k = 1, 2, \dots, m$ the following formula is valid:

$$\begin{aligned} &\theta_k(\mathbf{y}) = \max r_k \\ &\text{subject to} \\ &\quad r_k - y_i \leqslant M z_{ki}, \quad z_{ki} \in \{0, 1\} \\ &\qquad \text{for } i = 1, 2, \dots, m \\ &\quad \sum_{i=1}^{m} z_{ki} \leqslant k - 1 \end{aligned} \tag{9}$$

where M is a sufficiently large constant (larger than any possible difference between various individual outcomes y_i). Note that for $k = 1$ all the binary variables z_{1i} are enforced to 0 thus reducing the optimization to the standard LP model for that case.

The entire OWA optimization model (8) can be formulated as the following mixed integer multiple criteria problem:

$$\begin{aligned} &\max \quad \sum_{k=1}^{m} w_k r_k \\ &\text{subject to} \\ &\quad r_k - \mathbf{c}^i\mathbf{x} \leqslant M z_{ki} \quad \text{for } i = 1, 2, \dots, m \; k = 1, \dots, m \\ &\quad z_{ki} \in \{0, 1\} \qquad\quad \text{for } i = 1, 2, \dots, m \; k = 1, \dots, m \\ &\quad \sum_{i=1}^{m} z_{ki} \leqslant k - 1 \quad \text{for } k = 1, \dots, m \\ &\quad \mathbf{x} \in Q \end{aligned} \tag{10}$$

In this paper we focus our analysis on the OWA aggregations with the monotonic weights satisfying

$$w_1 > w_2 > \cdots > w_{m-1} > w_m > 0 \tag{11}$$

The OWA aggregation (7) has then the property of *equitability* (satisfies the principle of transfers)

$$\begin{aligned} &a(y_1, \dots, y_{i'} - \varepsilon, \dots, y_{i''} + \varepsilon, \dots, y_m) \\ &\quad > a(y_1, y_2, \dots, y_m) \quad \text{for } 0 < \varepsilon < y_{i'} - y_{i''} \end{aligned} \tag{12}$$

The equitability property guarantees that an equitable transfer of an arbitrarily small amount from the larger outcome to a smaller outcome results in more preferred achievement vector. For instance, when locating public facilities, we want to consider all the clients impartially and equally. Thus the distribution of distances (outcomes) among the clients is more important than the assignment of several distances (outcomes) to the specific clients. In other words, a location pattern generating individual distances: 4, 2 and 0 for clients 1, 2 and 3, respectively, should be considered equally good as a solution generating distances 0, 2 and 4. Moreover, according to the requirement of equal treatment of all clients a location pattern generating all distances equal to 2 should be considered better than both the above solutions [11]. In the stochastic problems uniform objectives represent various possible realizations of the same (uncertain) outcome under several scenarios and their equitability corresponds to the risk aversion [1]. Actually, an achievement vector represents then the discrete distribution of return defined as m-dimensional lottery, more equal returns implies a less risky lottery itself. In Section 4, we employ such a model of the portfolio optimization problem [12] to generate computational tests.

Maximization of aggregation functions satisfying the properties of strict monotonicity (4), impartiality (6) and equitability (12) generates the so-called equitably efficient solutions (cf. [7] for the formal axiomatic definition). In the case of strictly monotonic weights (11), every solution maximizing the OWA function is an equitably efficient solution of the original multiple criteria problem. Moreover, for linear multiple criteria problems, we consider, every equitably efficient solution can be found as an optimal solution to the OWA aggregation with appropriate strictly decreasing positive weights [7].

The OWA aggregation (7) is obviously a piecewise linear function since it remains linear within every area of the fixed order of arguments. Exactly, for any permutation τ of I, within the area of outcomes satisfying inequalities $y_{\tau(1)} \leqslant y_{\tau(2)} \leqslant \cdots \leqslant y_{\tau(m)}$, the following formula is valid:

$$\sum_{i=1}^{m} w_i \theta_i(\mathbf{y}) = \sum_{i=1}^{m} w_i y_{\tau(i)} = \sum_{i=1}^{m} w_{\tau^{-1}(i)} y_i \tag{13}$$

where τ^{-1} is the inverse of τ, i.e. $\tau^{-1}(\tau(i)) = i$ for $i = 1, 2, \ldots, m$. Moreover, if weights w_i are decreasing then the following inequality holds:

$$\sum_{i=1}^{m} w_i y_{\tau(i)} \geqslant \sum_{i=1}^{m} w_i \theta_i(\mathbf{y})$$

for any permutation τ of I. Hence, for any $0 \leqslant \lambda \leqslant 1$, one gets

$$\begin{aligned}\sum_{i=1}^{m} w_i \theta_i(\lambda \mathbf{y}' + (1-\lambda)\mathbf{y}'') &= \lambda \sum_{i=1}^{m} w_i y'_{\bar{\tau}(i)} + (1-\lambda) \sum_{i=1}^{m} w_i y''_{\bar{\tau}(i)} \\ &\geqslant \lambda \sum_{i=1}^{m} w_i \theta_i(\mathbf{y}') + (1-\lambda) \sum_{i=1}^{m} w_i \theta_i(\mathbf{y}'')\end{aligned}$$

where $\bar{\tau}$ denotes the permutation representing the Θ ordering of vector $\lambda \mathbf{y}' + (1-\lambda)\mathbf{y}''$. Thus the equitable OWA aggregations $a_{\mathbf{w}}(\mathbf{y})$ are concave functions of $\mathbf{y}$. While equal weights define the linear aggregation, several decreasing sequences of weights (11) lead us to various concave, piecewise linear, monotonic aggregation functions (see Fig. 1).

When differences among weights (11) tend to infinity, the OWA aggregation approximates the leximin ranking of the ordered outcome vectors [2,18]. That means, as the limiting case of the OWA problem (8), we get the lexicographic problem:

$$\operatorname{lexmax} \{(\theta_1(\mathbf{Cx}), \theta_2(\mathbf{Cx}), \ldots, \theta_m(\mathbf{Cx})) : \mathbf{x} \in Q\} \tag{14}$$

where first $\theta_1(\mathbf{Cx})$ is maximized, next $\theta_2(\mathbf{Cx})$ and so on. Problem (14) represents the lexicographic

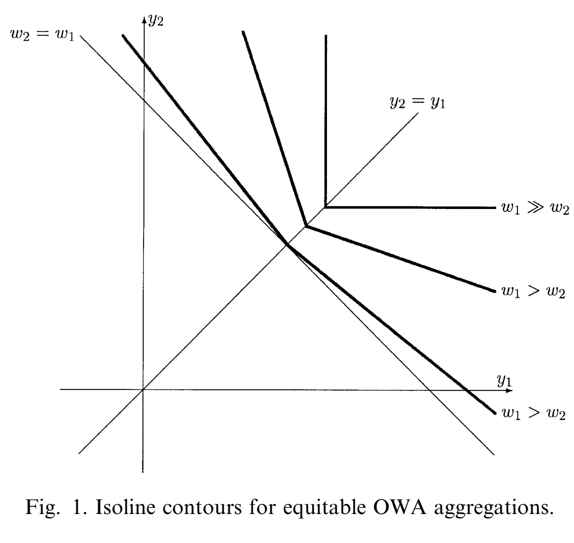

Fig. 1. Isoline contours for equitable OWA aggregations.

maximin approach to the original multiple criteria problem (1). It is a refinement (regularization) of the standard maximin optimization, but in the former, in addition to the smallest outcome, we maximize also the second smallest outcome (provided that the smallest one remains as large as possible), maximize the third smallest (provided that the two smallest remain as large as possible), and so on. Note that the lexicographic maximization is not applied to any specific order of the original criteria. Nevertheless, in the case of LP problems, there exists a dominating objective function which is constant on the entire optimal set of the maximin problem [9]. Hence, having solved the maximin problem, one may try to identify the dominating objective and eliminate it to formulate a restricted maximin problem on the former optimal set. Therefore, the lexicographic maximin solution to LP problems can be found by sequential maximin optimization with elimination of the dominating functions. This procedure is based, however, on the preemptive nature of the lexicographic optimization and cannot be extended to the OWA aggregation where some compensation between the ordered outcomes is allowed.

In the fuzzy methodologies the so-called 'andness' and 'orness' properties of the aggregation of the membership functions are considered [19] rather than the equitability. Note that the equitability property, we consider, corresponds to the 'and-like' character of the aggregation operator. Namely, taking into account the renumbering and the normalization of the weights we introduced, the andness measure (cf. [16]) takes the form $\sum_{i=1}^{m}((m-i)/(m-1))w_i/\sum_{i=1}^{m}w_i$, and inequality (11) guarantees that the measure is greater than 0.5.

3. LP formulations

3.1. Max–min model

The ordering operator Θ used in the OWA aggregation is nonlinear and, in general, it is hard to implement. However, with decreasing weights (11), as a concave piecewise linear function (13) the OWA aggregation can be expressed in the form:

$$a_{\mathbf{w}}(\mathbf{y}) = \min_{\tau\in\Pi}\left(\sum_{i=1}^{m} w_{\tau(i)}y_i\right) \tag{15}$$

where Π denotes the set of all permutations τ of I. Thus the OWA problem (8) is a max–min LP problem. This leads to the following LP formulation:

$$\max z \tag{16}$$

subject to

$$\mathbf{Ax} = \mathbf{b} \tag{17}$$

$$\mathbf{y} - \mathbf{Cx} = 0 \tag{18}$$

$$z - \sum_{i=1}^{m} w_{\tau(i)}y_i \leqslant 0 \quad \text{for } \tau\in\Pi \tag{19}$$

$$x_j \geqslant 0 \quad \text{for } j = 1,2,\ldots,n \tag{20}$$

This is an LP problem with $n+m+1$ variables and $p+m+m!$ constraints. That means the number of constraints is much larger than the number of variables. The huge number of constraints makes the usefulness of problem (16)–(20) questionable. However, all the inequalities (19) are defined by a single vector of weights w_i and they can be generated progressively during the solution process.

While solving an LP problem with the simplex method we prefer smaller number of constraints than variables since it results in a smaller dimension of basis and thereby in the lower computational complexity. Therefore, for the simplex approach it is much better to deal with the dual of (16)–(20) than with the original problem itself. Introducing the dual variables: $\mathbf{u} = (u_1,\ldots,u_p)$, $\mathbf{v} = (v_1,\ldots,v_m)$ and $\mathbf{t} = (t_\tau)_{\tau\in\Pi}$ corresponding to the constraints (17), (18) and (19), respectively, we get the following dual:

$$\min \mathbf{ub} \tag{21}$$

subject to

$$\mathbf{uA} - \mathbf{vC} \geqq \mathbf{0} \tag{22}$$

$$v_i - \sum_{\tau\in\Pi} w_{\tau(i)}t_\tau = 0 \quad \text{for } i = 1,2,\ldots,m \tag{23}$$

$$\sum_{\tau\in\Pi} t_\tau = 1 \tag{24}$$

$$t_\tau \geqslant 0 \quad \text{for } \tau\in\Pi \tag{25}$$

The dual problem (21)–(25) has $m!$ columns corresponding to variables t_τ. However, these columns can be handled implicitly with the column generation scheme. Note that each column corresponding to t_τ has the unit coefficient in row (24) and coefficients $-w_{\tau(i)}$ in rows (23). Thus there is no reason to keep them explicitly. We only need to identify the best column during the pricing and to generate the selected column for pivoting.

During the course of the simplex method, having the current basis $\mathbf{B}$ we have defined the current primal basic solution $(\mathbf{u}^0, \mathbf{v}^0, \mathbf{t}^0)$ and the current dual basic solution (the dual multipliers) $(\mathbf{x}^0, \mathbf{y}^0, z^0)$. The reduced cost for variable t_τ is given by the formula

$$d(t_\tau) = \sum_{i=1}^{m} w_{\tau(i)} y_i^0 - z^0 \quad \text{for } \tau \in \Pi$$

Due to (15), the solution to the pricing problem $\min_{\tau \in \Pi} d(t_\tau)$ is given by permutation $\bar{\tau}$ such that its inverse $\bar{\tau}^{-1}$ orders nondecreasingly $\mathbf{y}^0$, i.e.

$$y^0_{\bar{\tau}^{-1}(1)} \leqslant y^0_{\bar{\tau}^{-1}(2)} \leqslant \cdots \leqslant y^0_{\bar{\tau}^{-1}(m)} \tag{26}$$

where $\bar{\tau}^{-1}(\bar{\tau}(i)) = i$ for $i = 1, 2, \ldots, m$. In the case of all different coefficients in vector $\mathbf{y}^0$, there is a unique such a permutation $\bar{\tau}$ and the uniquely defined incoming column. When some coefficients are equal, then (26) defines a group of columns where the weights are permuted within the subsets of indices corresponding to equal coefficients y_i^0. Any such a single column (permutation) can be selected as the incoming one.

In the case of multiple permutations $\bar{\tau}$ solving (26), one may try to increase the simplex method efficiency by consideration of linear combinations (with positive scaling factors) of the columns corresponding to the alternative permutations. Such a combination may be used as an auxiliary incoming column. This approach is justified since the combined column corresponds to the combination of inequalities (19) that can be added to the primal without affecting the solution. However, in our initial computational testing we have not analyzed this modification.

When differences among weights (11) become large enough, then the OWA aggregation represents the leximin ranking of the ordered outcome vectors [18]. Implementation of such weights may cause, however, serious numerical difficulties. The max–min model can be extended to handle directly a limiting case of the lexicographic maximin problem (14). The resulting problem contains, however, a set of lexicographic inequality constraints replacing (19). Dual approach to such a problem is still possible [6], but it does not offer any simplification of the existing solution method [9].

3.2. Deviational model

An alternative LP formulation of the OWA optimization problem uses the cumulative ordered achievement vectors. Applying to ordered achievement vectors $\Theta(\mathbf{y})$ a linear cumulative map one gets

$$\bar{\theta}_k(\mathbf{y}) = \sum_{i=1}^{k} \theta_i(\mathbf{y}) \quad \text{for } k = 1, 2, \ldots, m \tag{27}$$

The quantities $\bar{\theta}_k(\mathbf{y})$ for $k = 1, 2, \ldots, m$ express, respectively: the worst (smallest) outcome, the total of the two worst outcomes, the total of the three worst outcomes, etc. When normalized by k the quantities $\bar{\theta}_k(\mathbf{y})/k$ can be interpreted as the worst conditional means.

The optimization formula (9) for $\theta_k(\mathbf{y})$ can easily be extended to define $\bar{\theta}_k(\mathbf{y})$. Namely, for any $k = 1, 2, \ldots, m$ the following formula is valid:

$$\bar{\theta}_k(\mathbf{y}) = \max kr_k - \sum_{i=1}^{m} d_{ki}$$

subject to

$$r_k - y_i \leqslant d_{ki}, d_{ki} \geqslant 0 \qquad \text{for } i = 1, 2, \ldots, m \tag{28}$$

$$d_{ki} \leqslant Mz_{ki}, z_{ki} \in \{0, 1\} \quad \text{for } i = 1, 2, \ldots, m$$

$$\sum_{i=1}^{m} z_{ki} \leqslant k - 1$$

where M is a sufficiently large constant. However, the optimization problem defining the cumulated ordered outcome can be dramatically simplified since all the binary variables (and the related constraints) turns out to be redundant as shown in the following theorem.

Theorem 1. *For any given vector* $\mathbf{y} \in R^m$, *the cumulated ordered coefficient* $\bar{\theta}_k(\mathbf{y})$ *can be found as the optimal value of the following LP problem*:

$$\bar{\theta}_k(\mathbf{y}) = \max kr_k - \sum_{i=1}^{m} d_{ki}$$
$$\text{subject to} \tag{29}$$
$$r_k - y_i \leqslant d_{ki}, d_{ki} \geqslant 0 \quad \text{for } i = 1, 2, \ldots, m$$

Proof. In order to prove the theorem we will show that the optimal value of problem (29) is the same as that of problem (28). First of all, let us notice that any feasible solution of (28) (when ignoring variables z_{ki}) is also feasible to problem (29). Moreover, such a solution has not more than $k-1$ positive values of variables d_{ki}. Opposite, every feasible solution of problem (29) corresponds to a feasible solution of problem (28), provided that it contains not more than $k-1$ positive values of variables d_{ki}. On the other hand, for any feasible solution to (29) which contains $s \geqslant k$ positive values of variables d_{ki} one can show an equally good or better alternative feasible solution with at most $s-1$ positive values of variables d_{ki}. Namely, by setting $\tilde{r}_k = r_k - \Delta$ and $\tilde{d}_{ki} = d_{ki} - \Delta$ for $d_{ki} > 0$, where $\Delta = \min_{d_{ki}>0} d_{ki}$ one gets

$$k\tilde{r}_k - \sum_{i=1}^{m} \tilde{d}_{ki} = kr_k - \sum_{i=1}^{m} d_{ki} + (s-k)\Delta \geqslant kr_k - \sum_{i=1}^{m} d_{ki}$$

Hence, the optimal value of problem (29) is the same as that of problem (28) which completes the proof. □

It follows from Theorem 1 that

$$\bar{\theta}_k(\mathbf{Cx}) = \max \left\{ kr_k - \sum_{i=1}^{m} d_{ik} : \mathbf{x} \in Q;\ \mathbf{c}^i\mathbf{x} \geqslant r_k - d_{ik},\ d_{ik} \geqslant 0, \quad \text{for } i = 1, 2, \ldots, m \right\}$$

or in a more compact form:

$$\bar{\theta}_k(\mathbf{Cx}) = \max \left\{ kr_k - \sum_{i=1}^{m} (\mathbf{c}^i\mathbf{x} - r_k)_+ : \mathbf{x} \in Q \right\}$$

where $(.)_+$ denotes the nonnegative part of a number and r_k is an auxiliary (unbounded) variable. The latter, with the necessary adaptation to the minimized outcomes in location problems, is equivalent to the computational formulation of the k-centrum model introduced in [13]. Hence, Theorem 1 provides an alternative proof of that formulation.

The ordered outcomes can be expressed as differences $\theta_i(\mathbf{y}) = \bar{\theta}_i(\mathbf{y}) - \bar{\theta}_{i-1}(\mathbf{y})$ for $i = 2, \ldots, m$ and $\theta_1(\mathbf{y}) = \bar{\theta}_1(\mathbf{y})$. Hence, the OWA problem (8) with weights w_i can be expressed in the form:

$$\min \left\{ \sum_{i=1}^{m} w'_i \bar{\theta}_i(\mathbf{Cx}) : \mathbf{x} \in Q \right\}$$

where coefficients w'_i are defined as $w'_m = w_m$ and $w'_i = w_i - w_{i+1}$ for $i = 1, 2, \ldots, m-1$. If the original weights w_i are monotonic (11), then $w'_i > 0$ for $i = 1, 2, \ldots, m$. This leads us to the following LP formulation of the OWA problem:

$$\max \quad \sum_{k=1}^{m} kw'_k r_k - \sum_{k=1}^{m} \sum_{i=1}^{m} w'_k d_{ik} \tag{30}$$

subject to

$$\mathbf{Ax} = \mathbf{b} \tag{31}$$
$$\mathbf{y} - \mathbf{Cx} = \mathbf{0} \tag{32}$$
$$d_{ik} \geqslant r_k - y_i \quad \text{for } i, k = 1, 2, \ldots, m \tag{33}$$
$$d_{ik} \geqslant 0 \quad \text{for } i, k = 1, 2, \ldots, m$$
$$x_j \geqslant 0 \quad \text{for } j = 1, 2, \ldots, n \tag{34}$$

It is an LP problem with $m^2 + 2m + n$ variables and $m^2 + m + p$ constraints. Thus, for many problems with not too large number of criteria m, say up to m representing a few dozens, problem (30)–(34) can be solved directly.

The number of constraints in problem (30)–(34) is similar to the number of variables. Nevertheless, for the simplex approach it may be better to deal with the dual of (30)–(34) than with the original problem. Note that variables d_{ik} in the primal are represented with singleton columns. Hence, the corresponding m^2 rows in the dual represent only simple upper bounds.

Introducing the dual variables: $\mathbf{u} = (u_1, \ldots, u_p)$, $\mathbf{v} = (v_1, \ldots, v_m)$ and $\mathbf{q} = (q_{ik})_{i,k=1,\ldots,m}$ corresponding to the constraints (31), (32) and (33), respectively, we get the following dual:

$$\begin{aligned} & \min\ \mathbf{ub} \\ & \text{subject to} \\ & \quad \mathbf{uA} - \mathbf{vC} \geqq \mathbf{0} \\ & \quad v_i - \sum_{k=1}^{m} q_{ik} = 0 \quad \text{for } i = 1,2,\ldots,m \\ & \quad \sum_{i=1}^{m} q_{ik} = k w'_k \quad \text{for } k = 1,2,\ldots,m \\ & \quad 0 \leqslant q_{ik} \leqslant w'_k \quad \text{for } i,k = 1,2,\ldots,m \end{aligned} \tag{35}$$

The dual problem (35) contains: $2m + n$ structural constraints, $p + m$ unbounded variables and m^2 bounded variables. Hence, the dual can be directly solved for a quite large number m.

One may notice that the columns corresponding to m^2 variables q_{ik} forms the transportation/assignment matrix. This opens an opportunity to employ special techniques of the simplex SON algorithm [4] for implicit handling of these variables. Such techniques increase dramatically efficiency of the simplex method but they require a special tailored implementation. Therefore, we have not tested this approach within our initial computational experiments based on the use of a general purpose LP code.

An additional advantage of the deviational model is related to its capability of direct application to the limiting case representing the leximin ranking of the ordered outcome vectors [2,18]. Due to the preemptive nature of the lexicographic optimization, maximization of the second smallest outcome can be replaced by maximization of the total of two smallest outcomes (provided that the smallest one remains as large as possible), and so on. Hence, by (27), the lexicographic maximin problem (14) is equivalent to the problem

$$\text{lexmax } \{(\bar{\theta}_1(\mathbf{Cx}), \bar{\theta}_2(\mathbf{Cx}), \ldots, \bar{\theta}_m(\mathbf{Cx})) : \mathbf{x} \in Q\}$$

Following Theorem 1, the above leads us to a standard lexicographic optimization problem with predefined linear criteria:

$$\text{lexmax}\left(r_1 - \sum_{i=1}^{m} d_{i1}, 2r_2 - \sum_{i=1}^{m} d_{i2}, \ldots, m r_m - \sum_{i=1}^{m} d_{im} \right)$$

subject to: (31)–(34)

Note that this direct lexicographic formulation remains valid for nonconvex (e.g. discrete) feasible sets Q, where the standard sequential approaches [8,9] are not applicable [10].

4. Computational tests

We have run initial tests to analyze the computational performances of the max–min and the deviational models. For this purpose we have solved randomly generated problems with varying number of decision variables n and number of criteria m while the basic LP feasible set has been defined by a single knapsack-type constraint. Such problems may be interpreted as portfolio selection decisions according to the (discrete) scenario analysis approach [12] as described in the following example.

Example 1. Consider a simple problem of portfolio optimization. Let $J = \{1, 2, \ldots, n\}$ denote the set of securities in which one intends to invest a capital. We assume, as usual, that for each security $j \in J$ there is given a vector of data $(c_{ij})_{i=1,2,\ldots,m}$, where c_{ij} is the observed (or forecasted) rate of return of security j under scenario i (hereafter referred to as outcome). We consider discrete distributions of returns defined by the finite set $I = \{1, 2, \ldots, m\}$ of equally probable scenarios. The outcome data forms an $m \times n$ matrix $\mathbf{C} = (c_{ij})_{i=1,\ldots,m;\ j=1,\ldots,n}$ which columns correspond to securities while rows $\mathbf{c}_i = (c_{ij})_{j=1,2,\ldots,n}$ correspond to outcomes. Further, let $\mathbf{x} = (x_j)_{j=1,2,\ldots,n}$ denote the vector of decision variables defining a portfolio. Each variable x_j expresses the portion of the capital invested in the corresponding security. Portfolio $\mathbf{x}$ generates outcomes

$$\mathbf{y} = \mathbf{Cx} = (\mathbf{c}_1\mathbf{x}, \mathbf{c}_2\mathbf{x}, \ldots, \mathbf{c}_m\mathbf{x})$$

The portfolio selection problem can be considered as an LP optimization problem with m uniform objective functions $f_i(\mathbf{x}) = \mathbf{c}_i\mathbf{x} = \sum_{j=1}^{n} c_{ij}x_j$ to be maximized [12]:

$$\max \left\{ \mathbf{Cx} : \sum_{j=1}^{n} x_j = 1,\ x_j \geqslant 0 \text{ for } j = 1,2,\ldots,n \right\}$$

Hence, our portfolio optimization problem can be considered a special case of the multiple criteria problem (1) and one may seek an optimal portfolio with the OWA aggregation

$$\max \left\{ \sum_{i=1}^{m} w_i \theta_i(\mathbf{Cx}) : \sum_{j=1}^{n} x_j = 1, \right.$$
$$\left. x_j \geqslant 0 \text{ for } j = 1, 2, \ldots, n \right\} \quad (36)$$

Note that the aggregation must be equitable to model risk averse preferences [1]. Hence, the weights w_i have to be monotonic in the sense of (11).

Our computational tests were based on the randomly generated problems (36). The generation procedure worked as follows. First, for each security j the maximum rate of return r_j was generated as a random number uniformly distributed in the interval [0.05, 0.15]. Next, this value was used to generate specific outcomes c_{ij} (the rate of return under scenarios i) as random variables uniformly distributed in the interval $[-0.75r_j, r_j]$. Further, strictly decreasing and positive weights w_i (11) were generated. The weights were not normalized which allowed us to define them by the corresponding increments $\delta_i = w_i - w_{i-1}$. The latter were generated as uniformly distributed random values in the range of [1.0, 2.0], except from a few (5 on average) possibly larger increments ranged from 1.0 to $m/3$.

We tested solution times for different size parameters m and n. For each number of decision variables (securities) n and number of criteria (scenarios) m we solved 20 randomly generated problems (36). All computations were performed on a PC with the Pentium 200 MHz processor employing the CPLEX 6.0 package [5]. The 500 seconds time limit was used in all the computations.

Table 1 presents the solution times for the max–min model solved by the column generation technique. Exactly, the max–min dual model (21)–(25) was first simplified by eliminating variables v and thus making it less dependent on the number of criteria (scenarios) m. The resulting problem was solved by the column generation technique with the use of the CPLEX callable library [5]. The solution times report the averages of 20 randomly generated problems. Numbers in parentheses, next to the time values, show the numbers of tests among 20 that ended up either with numerical difficulties or with timeout (of 500 seconds). The empty cell (minus sign) shows that the timeout occurred for all 20 instances to be solved. One may notice that despite the tremendous problem size with respect to m (m! columns), the column generation technique allowed us to solve quite large instances with m up to 300 in a reasonable time, provided that the number of structural variables n was rather limited.

In Tables 2 and 3 we show the solution times for the primal (30)–(34) and the dual (35) forms of the deviational model, respectively. Both forms

Table 1
Solution times for the max–min model: the column generation approach

Number of criteria (m)	Number of variables (n)							
	5	10	20	40	60	100	140	200
10	0.05	0.10	0.10	0.20	0.20	0.40	0.50	1.00
20	0.10	0.15	0.30	0.55	0.90	1.60 (1)	2.50	3.75
30	0.15	0.25	0.45	1.60	2.55	4.55 (1)	7.15	11.45
40	0.15	0.35	0.75	2.35	4.65	9.70	14.25	28.70
60	0.15	0.60	1.85	5.25	10.05	20.15 (1)	34.60 (1)	73.80 (3)
100	0.30	1.10	3.75	15.45	22.35	60.80 (2)	80.15 (9)	175.50 (12)
140	0.40	1.65	5.15	25.50 (2)	48.20 (2)	68.90 (9)	–	–
200	0.50	1.95	9.50	36.25 (4)	50.20 (8)	–	–	–
300	0.65	4.55	16.75	78.35 (4)	–	–	–	–

Table 2
Solution times for the deviational model: the primal approach

Number of criteria (m)	Number of variables (n)							
	5	10	20	40	60	100	140	200
10	0.10	0.15	0.15	0.15	0.20	0.20	0.25	0.35
20	0.70	0.80	1.00	1.20	1.25	1.45	1.65	2.35
30	3.20	3.80	4.05	4.80	4.95	6.05	6.50	7.55
40	9.70	11.90	13.70	15.90	18.00	20.00	20.05	23.60
60	58.05	69.35	80.95	90.35	106.55	113.95	127.05	139.25
100	–	–	–	–	–	–	–	–

Table 3
Solution times for the deviational model: the dual approach

Number of criteria (m)	Number of variables (n)							
	5	10	20	40	60	100	140	200
10	0.05	0.10	0.10	0.15	0.15	0.20	0.25	0.30
20	0.30	0.35	0.40	0.60	0.75	1.00	1.20	1.70
30	0.80	1.00	1.55	2.15	2.65	3.35	4.30	6.20
40	1.95	2.35	3.20	5.25	6.75	9.50	11.80	16.80
60	7.30	8.80	10.95	20.75	31.30	44.95	55.70	65.25
100	49.05	54.60	65.40	104.15	173.10	278.80	–	–
140	191.00	196.50	226.10	–	–	–	–	–
200	–	–	–	–	–	–	–	–

were directly solved by the CPLEX code without taking advantages of the constraints structure specificity. One can see the primal model performing very well for problems with a limited number of criteria (m up to 60). While increasing m, the solution times raise even faster than one may expect according to m^2 impact on the problem size. On the other hand, in contrary to the max–min model, the deviational model is not very much sensitive on the number of structural variables n. The solution times for the deviational model turn out to be much more stable among 20 instances randomly generated for each table cell. In particular, we have not noticed specific instances ending with the numerical difficulties or the solution times much longer than the average (reported in the tables).

Table 3 shows that switching to the dual formulations of the deviational model significantly reduces the computational times. Problems with up to 60 criteria are solved in less than 100 seconds while problems with up to 100 criteria may be solved within the time limit of 500 seconds. Similarly to the primal formulation, the solution times raise very fast with increasing number of criteria. However, the number of 100 criteria seems to be quite enough for most applications, including the fuzzy aggregations and decisions under risk, while Table 3 shows roughly linear dependence of the solution times on the number of structural variables (n). Recall also that the problems have been solved directly by the CPLEX code without taking advantages of the constraints structure specificity. Taking advantages of the LP embedded network structure by the techniques of the simplex SON algorithm [4] should further reduce the solution times allowing to solve effectively larger problems (35).

5. Concluding remarks

The problem of aggregating multiple criteria to form overall objective functions is of considerable importance in many disciplines. A primary factor in the determination of the structure of such aggregation is the relationship between the criteria involved. There are several decision problems where the multiple criteria are uniform and need to be treated impartially. Moreover, the equity among the criteria is an important issue.

The weighted sum aggregation violates the requirement of impartiality as it assigns the weights to the specific criteria. Yager [16] introduced the so-called OWA aggregation where the weights are assigned to the ordered values rather than to the specific criteria. The equitability properties correspond to the and-like character of the aggregation operator and they are guaranteed by the monotonic weights.

The ordering operator used to define the OWA aggregation is, in general, hard to implement. We have shown that the OWA aggregations with the monotonic weights can be modeled by introducing auxiliary linear constraints. Hence, an LP problem with the OWA objective can be formed as a standard linear program. Two alternative LP formulations have been introduced. While the max–min model requires the column generation techniques to overcome its huge number of columns, the deviational model can be solved directly as introducing only m^2 variables. Moreover, the deviational model can be directly applied to the lexicographic maximin optimization.

Initial computational experiments show that both formulations enable to solve effectively medium size problems. Actually, the number of 100 criteria covered by the dual approach to the deviational model seems to be quite enough for most applications, including the fuzzy aggregations and decisions under risk. The problems have been solved directly by general purpose LP code while taking advantages of the constraints structure specificity may remarkably extend the solution capabilities. In particular, the simplex SON algorithm [4] may be used for exploiting the LP embedded network structure in the dual form of the deviational model. This seems to be a very promising direction for further research.

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