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European Journal of Operational Research 122 (2000) 374–391

EUROPEAN
JOURNAL
OF OPERATIONAL
RESEARCH

www.elsevier.com/locate/orms

Inequality measures and equitable approaches to location problems [☆]

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Received 1 October 1998; accepted 1 April 1999

Abstract

Location problems can be considered as multiple criteria models where for each client (spatial unit) there is defined an individual objective function, which measures the effect of a location pattern with respect to the client satisfaction (e.g., it expresses the distance or travel time between the client and the assigned facility). This results in a multiple criteria model taking into account the entire distribution of individual effects (distances). Moreover, the model enables us to introduce the concept of equitable efficiency which links location problems with theories of inequality measurement. In this paper special attention is paid to solution concepts based on the bicriteria optimization of the mean distance and the absolute inequality measures. The restrictions for the trade-offs are identified which guarantee that the bicriteria approaches comply with the concept of equitable efficiency. These results are further generalized to bicriteria approaches not using directly the trade-off technique. © 2000 Elsevier Science B.V. All rights reserved.

Keywords: Location; Multiple criteria; Efficiency; Equity

1. Introduction

Public goods and services are typically provided and managed by governments in response to perceived and expressed need. The spatial distribution of public goods and services is influenced by facility location decisions. The generic location problem that we consider may be stated as follows. There is given a set $I = \{1, 2, \dots, m\}$ of m clients

(service recipients). Each client is represented by a specific point in the geographical space. There is also given a set Q of location patterns (location decisions). For each client i ($i \in I$) a function $f_i(\mathbf{x})$ of the location pattern \mathbf{x} has been defined. This function, called the individual objective function, measures the outcome (effect) $y_i = f_i(\mathbf{x})$ of the location pattern for client i (Marsh and Schilling, 1994). In the simplest problems an outcome usually expresses the distance. However, we emphasize to the reader that we do not restrict our considerations to the case of outcomes measured as distances. They can be measured (modeled) as travel time, travel costs as well as in a more

[☆] This work was partially supported by The State Committee for Scientific Research under grant KBN-8T11A02411.

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subjective way as relative travel costs (e.g., travel costs by client incomes) or ultimately as the levels of client dissatisfaction (individual disutility) of locations. In typical formulations of location problems related to desirable facilities a smaller value of the outcome (distance) means a better effect (higher service quality or client satisfaction). This remains valid for location of obnoxious facilities if the distances are replaced with their complements to some large number. Therefore, without loss of generality, we can assume that all outcomes are nonnegative ($y_i \geq 0$) and each individual outcome y_i is to be minimized.

Frequently, one may be interested in putting into location model some additional client weights $v_i > 0$ to represent the service demand. Integer weights can be interpreted as numbers of unweighted clients located at exactly the same place (with distances 0 among them). For theoretical considerations we will assume that the problem is transformed (disaggregated) to the unweighted one (that means all the client weights are equal to 1). Note that such a disaggregation is possible for integer as well as rational client weights, but it usually dramatically increases the problem size. Therefore, we are interested in solution concepts which can be applied directly to the weighted problem. While discussing such solution concepts we will use the normalized client weights,

$$\bar{v}_i = v_i / \sum_{i=1}^m v_i \quad \text{for } i = 1, 2, \dots, m,$$

rather than the original quantities v_i . Note that, in the case of unweighted problem (all $v_i = 1$), all the normalized weights are given as $\bar{v}_i = 1/m$.

A host of operational models has been developed to deal with facility location optimization (cf. Love et al., 1988; Francis et al., 1992; Current et al., 1990). Most classical location studies focus on the minimization of the mean (or total) distance (the median concept) or the minimization of the maximum distance (the center concept) to the service facilities (Morrill and Symons, 1977). Both the median and the center solution concepts are well defined for aggregated location models using client weights $v_i > 0$ to represent several clients (service demand) at the same geographical point.

Exactly, for the weighted location problem, the *center* solution concept is defined by the optimization problem

$$\min \left\{ \max_{i=1, \dots, m} f_i(\mathbf{x}) : \mathbf{x} \in Q \right\} \quad (1)$$

and it is not affected by the client weights at all. The *median* solution concept is defined by the optimization problem

$$\min \left\{ \sum_{i=1}^m \bar{v}_i f_i(\mathbf{x}) : \mathbf{x} \in Q \right\}. \quad (2)$$

In the above problem the objective function is defined as the *mean* (average) outcome

$$\mu(\mathbf{y}) = \sum_{i=1}^m \bar{v}_i y_i$$

but the problem (2) itself is also equivalent to minimization of the total outcome $\sum_{i=1}^m y_i$. Both concepts minimize only simple scalar characteristics of the distribution: the maximal (the worst) outcome and the mean outcome, respectively. In this paper all the outcomes (distances) for the individual clients are considered as the set of multiple uniform criteria to be minimized. This results in a multiple criteria model taking into account the entire distribution of distances. Moreover, the model enables us to link location problems with theories of inequality measurement (in particular the Pigou–Dalton approach) (Sen, 1973).

While locating public facilities, the issue of equity is becoming important. Equity is, essentially, an abstract socio-political concept that implies fairness and justice (Young, 1994). Nevertheless, equity is usually quantified with the so-called inequality measures to be minimized. Inequality measures were primarily studied in economics (Sen, 1973). However, Marsh and Schilling (1994) describe twenty different measures proposed in the literature to gauge the level of equity in facility location alternatives. The simplest inequality measures are based on the absolute measurement of the spread of outcomes, like the *mean (absolute) difference*

$$D(\mathbf{y}) = \frac{1}{2} \sum_{i=1}^m \sum_{j=1}^m |y_i - y_j| \bar{v}_i \bar{v}_j \quad (3)$$

or the *maximum (absolute) difference*

$$R(\mathbf{y}) = \frac{1}{2} \max_{i,j=1,\dots,m} |y_i - y_j|. \quad (4)$$

In the location framework better intuitive appeal may have inequality measures related to deviations from the mean outcome (Mulligan, 1991) like the *mean (absolute) deviation*

$$\delta(\mathbf{y}) = \sum_{y_i \geq \mu(\mathbf{y})} (y_i - \mu(\mathbf{y})) \bar{v}_i = \frac{1}{2} \sum_{i=1}^m |y_i - \mu(\mathbf{y})| \bar{v}_i \quad (5)$$

or the *maximum (upperside) deviation*

$$\Delta(\mathbf{y}) = \max_{i=1,\dots,m} (y_i - \mu(\mathbf{y})). \quad (6)$$

In economics one usually considers relative inequality measures normalized by mean outcome. Among many inequality measures perhaps the most commonly accepted by economists is the *Gini coefficient*, which has been recently also analyzed in the location context (Mulligan, 1991; Erkut, 1993). The Gini coefficient is the relative mean difference (Kendall and Stuart, 1958):

$$G(\mathbf{y}) = \frac{D(\mathbf{y})}{\mu(\mathbf{y})} = \frac{\sum_{i=1}^m \sum_{j=1}^m |y_i - y_j| \bar{v}_i \bar{v}_j}{2 \sum_{i=1}^m y_i \bar{v}_i}. \quad (7)$$

It can be relatively easily introduced into the location models with tools of linear programming (Mandell, 1991). Similarly, one may consider the relative mean deviation which is known as the *Schutz index* (Schutz, 1951).

One can easily notice that direct minimization of typical inequality measures (especially relative ones) contradicts the minimization of individual outcomes. As noticed by Erkut (1993), it is rather a common flaw of all the relative inequality measures that while moving away from the spatial units to be serviced one gets better values of the measure as the relative distances become closer to one another. As an extreme, one may consider an unconstrained continuous (single-facility) location problem and find that the facility located at (or

near) infinity will provide (almost) perfectly equal service (in fact, rather lack of service) to all the spatial units. Nevertheless, we show in this paper that some absolute inequality measures, namely: the maximum deviation (6), the mean deviation (5) and the mean difference (3), can be effectively used to incorporate equity factors into facility location decision models.

The paper is organized as follows. In the next section we introduce the equitable multiple criteria location model with the preference structure that complies with both the efficiency (Pareto-optimality) principle and with the Pigou–Dalton principle of transfers. Further, in Section 3, the concept of equitably efficient location patterns is formalized and there are developed general generation techniques based on the standard multiple criteria optimization applied to the cumulative ordered outcomes. Section 4 contains the main results showing that, under the assumption of bounded trade-offs, the bicriteria mean/equity approaches for selected absolute inequality measures (maximum deviation, mean deviation or mean difference) comply with the rules of equitable multiple criteria optimization. These results are, in Section 5, further generalized to bicriteria approaches not using directly the trade-off technique.

2. The model

Assuming that the generic location problem that we consider has been disaggregated to the unweighted form (all $v_i = 1$), it may be stated as the following multiple criteria minimization problem:

$$\min \{\mathbf{f}(\mathbf{x}) : \mathbf{x} \in Q\}, \quad (8)$$

where:

$\mathbf{f} = (f_1, \dots, f_m)$ is a vector-function that maps the decision space $X = \mathbb{R}^n$ into the outcome space $Y = \mathbb{R}^m$,

$Q \subset X$ denotes the feasible set of location patterns,

$\mathbf{x} \in X$ denotes the vector of decision variables (the location pattern).

A wide gamut of location problems can be considered within the framework of model (8).

The following example illustrates how a typical discrete location problem (Mirchandani and Francis, 1990) can be modeled in the form (8).

Example 1. In a typical discrete location problem there is given a set of m clients and a set of n potential locations for the facilities. Further, the number (or the maximal number) p of facilities to be located is given ($p \leq n$). The main decisions to be made can be described with the binary variables x_j ($j = 1, 2, \dots, n$) equal to 1 if location j is to be used and equal to 0 otherwise. To meet the problem requirements, the decision variables x_j have to satisfy the following constraints:

$$\sum_{j=1}^n x_j = p, \quad x_j \in \{0, 1\} \text{ for } j = 1, 2, \dots, n, \quad (9)$$

where the equation is replaced with the inequality (\leq) if p specifies the maximal number of facilities to be located. Note that constraints (9) take a very simple form of the binary knapsack problem with all the constraint coefficients equal to 1. However, for most location problems the feasible set has a more complex structure due to explicit consideration of allocation decisions. These decisions are usually modeled with the additional allocation variables x'_{ij} ($i = 1, 2, \dots, m; j = 1, 2, \dots, n$) equal to 1 if location j is used to service client i and equal to 0 otherwise. The allocation variables have to satisfy the following constraints:

$$\sum_{j=1}^n x'_{ij} = 1 \text{ for } i = 1, 2, \dots, m, \quad (10)$$

$$x'_{ij} \leq x_j \text{ for } i = 1, 2, \dots, m; j = 1, 2, \dots, n, \quad (11)$$

$$x'_{ij} \in \{0, 1\} \text{ for } i = 1, 2, \dots, m; j = 1, 2, \dots, n. \quad (12)$$

In the capacitated location problem the capacities of the potential facilities are given which implies some additional constraints.

Let $d_{ij} \geq 0$ ($i = 1, 2, \dots, m; j = 1, 2, \dots, n$) express the distance between client i and location j (or other effect of allocation client i to location j). For the standard uncapacitated location problem it is assumed that all the potential facilities provide the same type of service and each client is serviced

by the nearest located facility. The individual objective functions then take the following form:

$$f_i(\mathbf{x}) = \min_{j=1, \dots, n} \{d_{ij} : x_j = 1\} \text{ for } i = 1, 2, \dots, m.$$

With the explicit use of the allocation variables and the corresponding constraints (10) and (11) the individual objective functions f_i can be written in the linear form:

$$f_i(\mathbf{x}) = \sum_{j=1}^n d_{ij}x'_{ij} \text{ for } i = 1, 2, \dots, m. \quad (13)$$

These linear functions of the allocation variables are applicable for the uncapacitated as well as for the capacitated facility location problems.

In the case of location of desirable facilities a smaller value of the individual objective function means a better effect (smaller distance). This remains valid for location of obnoxious facilities if the distance coefficients are replaced with their complements to some large number: $d'_{ij} = d - d_{ij}$, where $d > d_{ij}$ for all $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$. Therefore, we can assume that each function f_i is to be minimized as stated in the multiple criteria problem (8).

We do not assume any special form of the feasible set while analyzing properties of the solution concepts. We rather allow the feasible set to be a general, possibly discrete (nonconvex), set. Similarly, we do not assume any special form of the individual objective functions nor their special properties (like convexity) while analyzing properties of the solution concepts. Therefore, the results of our analysis apply to various location problems.

Model (8) only says that we are interested in the minimization of all objective functions f_i for $i \in \{I = 1, 2, \dots, m\}$. In order to make it operational, one needs to assume some solution concept specifying what it means to minimize multiple objective functions. Vector-function \mathbf{f} maps the feasible set Q (as a subset of the decision space) into the outcome space Y . The elements of the outcome space we refer to as achievement vectors. An achievement vector $\mathbf{y} \in Y$ is attainable if it

expresses outcomes of a feasible solution $\mathbf{x} \in Q$ ($\mathbf{y} = \mathbf{f}(\mathbf{x})$).

Typical solution concepts for the location problems are based on some scalar measures of the achievement vectors. However, there are some concepts, like the lexicographic center (Ogryczak, 1997), which do not introduce directly any scalar measure, even though they rank the achievement vectors with a complete preorder. Therefore, we prefer to focus our analysis of solution concepts on the properties of the corresponding preference model. We assume that solution concepts depend only on evaluation of the achievement vectors and they do not take into account other solution properties not represented within achievement vectors. In fact, to the extent of our knowledge, all the solution concepts for location problems present in the literature satisfy this assumption. Thus, we can limit our considerations to the preference model in the outcome space Y .

The preference model is completely characterized by the relation of weak preference (Vincke, 1992), denoted hereafter with \preceq . Namely, we say that achievement vector $\mathbf{y}' \in Y$ is (strictly) preferred to $\mathbf{y}'' \in Y$ ($\mathbf{y}' \prec \mathbf{y}''$) iff $\mathbf{y}' \preceq \mathbf{y}''$ and $\mathbf{y}'' \not\preceq \mathbf{y}'$. Similarly, we say that achievement vector $\mathbf{y}' \in Y$ is indifferent or equally preferred to $\mathbf{y}'' \in Y$ ($\mathbf{y}' \cong \mathbf{y}''$) iff $\mathbf{y}' \preceq \mathbf{y}''$ and $\mathbf{y}'' \preceq \mathbf{y}'$. If a solution concept is defined by the minimization of some scalar function $g(\mathbf{y})$, then the corresponding preference model is defined by the relation: $\mathbf{y}' \preceq \mathbf{y}''$ iff $g(\mathbf{y}') \leq g(\mathbf{y}'')$.

The standard preference model related to the Pareto-optimal solution concept assumes that the preference relation \preceq is *reflexive*,

$$\mathbf{y} \preceq \mathbf{y}, \quad (14)$$

transitive,

$$(\mathbf{y}' \preceq \mathbf{y}'' \text{ and } \mathbf{y}'' \preceq \mathbf{y}''') \Rightarrow \mathbf{y}' \preceq \mathbf{y}''', \quad (15)$$

and *strictly monotonic*,

$$\mathbf{y} - \varepsilon \mathbf{e}_i \prec \mathbf{y} \quad \text{for } \varepsilon > 0, \quad i = 1, 2, \dots, m, \quad (16)$$

where \mathbf{e}_i denotes the i th unit vector in the criterion space. The last assumption expresses that for each individual objective function less is better (minimization), i.e., in terms of typical location prob-

lems, for each spatial unit, closer to the service means better.

The preference relations satisfying axioms (14)–(16) are called hereafter *rational preference relations*. The rational preference relations allow us to formalize the Pareto-optimal solution concept with the following definitions. We say that achievement vector \mathbf{y}' *rationally dominates* \mathbf{y}'' ($\mathbf{y}' \prec_r \mathbf{y}''$), iff $\mathbf{y}' \prec \mathbf{y}''$ for all rational preference relations \preceq . We say that a location pattern $\mathbf{x} \in Q$ is an *efficient (Pareto-optimal) solution* of the multiple criteria problem (8), iff $\mathbf{y} = \mathbf{f}(\mathbf{x})$ is rationally nondominated.

The relation of rational dominance may be expressed in terms of the vector inequality. As a consequence we can state that a location pattern $\mathbf{x}^0 \in Q$ is an efficient solution of the multiple criteria problem (8), if and only if, there does not exist $\mathbf{x} \in Q$ such that $f_i(\mathbf{x}) \leq f_i(\mathbf{x}^0)$ for all $i \in I$ where at least one strict inequality holds. The latter refers to the commonly used definition of the efficient solutions as feasible solutions for which one cannot improve any criterion without worsening another (e.g., Steuer, 1986). However, the axiomatic definition of the rational preference relation allows us to introduce additional properties of the preferences related to the principles of equity.

The concept of Pareto-optimal solutions is built for typical multiple criteria problems where values of the individual objective functions are assumed to be incomparable (Steuer, 1986). The individual objective functions in our multiple criteria location model express the same quantity (usually the distance) for various clients. Thus, the functions are uniform in the sense of the scale used and their values are directly comparable. This is ultimately true for all location models as long as the modeler is capable to express the individual outcomes (and the outcome coefficients d_{ij}) in the unique scale of client dissatisfaction (disutility). Moreover, especially when locating public facilities, we want to consider all the clients impartially and equally. Thus, the distribution of distances (outcomes) among the clients is more important than the assignment of several distances (outcomes) to the specific clients. In other words, a location pattern generating individual distances: 4, 2 and 0 for cli-

ents 1, 2 and 3, respectively, should be considered equally good as a solution generating distances 0, 2 and 4. Moreover, according to the requirement of equal treatment of all clients a location pattern generating all distances equal to 2 should be considered better than both the above solutions.

For multiple criteria problems with uniform and equally important objective functions we introduce an efficiency concept based rather on the distribution of outcomes than on the achievement vectors themselves. For this purpose, we assume that the preference model satisfies the principle of impartiality (anonymity)

$$(y_{\tau(1)}, y_{\tau(2)}, \dots, y_{\tau(m)}) \cong (y_1, y_2, \dots, y_m) \tag{17}$$

for any $\tau \in \Pi(I)$,

where $\Pi(I)$ is the set of all permutations of the set I . Condition (17) means that any permutation of the achievement vector is equally good (indifferent) as the original achievement vector. Adding the principle of impartiality to the domination relation leads us to the concept of symmetric domination which is not affected by any permutation of the achievement vector coefficients (Ogryczak, 1999).

While locating public facilities, the preference model should take into account equity of the effects (distances). According to the theory of equity measurement (Sen, 1973; Allison, 1978), the preference model should satisfy the (Pigou–Dalton) principle of transfers. The principle of transfers states that a transfer of small amount from an outcome to any relatively worse-off outcome results in a more preferred achievement vector. As a property of the preference relation, the principle of transfers takes the form of the following axiom:

$$y_{i'} > y_{i''} \Rightarrow \mathbf{y} - \varepsilon \mathbf{e}_{i'} + \varepsilon \mathbf{e}_{i''} \prec \mathbf{y}$$

for $0 < \varepsilon < y_{i'} - y_{i''}$; $i', i'' \in I$. (18)

Requirement of impartiality (17) and the principle of transfers (18) do not contradict the multiple criteria optimization axioms (14)–(16). Therefore, we can consider equitable multiple criteria optimization (Kostreva and Ogryczak, 1999a) based on the *equitable rational preference relations* defined by axioms (14)–(18). The equitable rational preference relations allow us to define the concept

of equitably efficient solution, similar to the standard efficient (Pareto-optimal) solution defined with the rational preference relations. We say that achievement vector \mathbf{y}' *equitably dominates* \mathbf{y}'' ($\mathbf{y}' \prec_e \mathbf{y}''$), iff $\mathbf{y}' \prec \mathbf{y}''$ for all equitable rational preference relations \preceq . We say that a location pattern (feasible solution) $\mathbf{x} \in Q$ is *equitably efficient* (is an equitably efficient solution of the multiple criteria problem (8)), if and only if there does not exist any $\mathbf{x}' \in Q$ such that $\mathbf{f}(\mathbf{x}') \prec_e \mathbf{f}(\mathbf{x})$. Note that each equitably efficient solution is also an efficient solution but not vice versa.

Scale invariance is widely considered an additional axiom for equity measurement. We say that a preference relation \preceq is *scale invariant* (satisfies the principle of scale invariance) if for any achievement vectors $\mathbf{y}', \mathbf{y}'' \in Y$ and for any positive constant c ,

$$\mathbf{y}' \preceq \mathbf{y}'' \Rightarrow c\mathbf{y}' \preceq c\mathbf{y}''.$$

We do not assume the principle of scale invariance as an axiom for the preference model. Nevertheless, we pay attention if solution concepts comply with it as such a principle is important for maintaining stability of the solution, and for creating well-defined models. In fact, all the concepts discussed here comply with the principle of scale invariance.

3. Equitably efficient solutions

The relation of equitable dominance can be expressed as a vector inequality on the cumulative ordered achievement vectors. This can be mathematically formalized as follows. First, we introduce the ordering map $\Theta : \mathbb{R}^m \rightarrow \mathbb{R}^m$ such that

$$\Theta(\mathbf{y}) = (\theta_1(\mathbf{y}), \theta_2(\mathbf{y}), \dots, \theta_m(\mathbf{y})),$$

where

$$\theta_1(\mathbf{y}) \geq \theta_2(\mathbf{y}) \geq \dots \geq \theta_m(\mathbf{y})$$

and there exists a permutation τ of set I such that $\theta_i(\mathbf{y}) = y_{\tau(i)}$ for $i = 1, 2, \dots, m$. This allows us to focus on distributions of outcomes impartially. Next, we apply to ordered achievement vectors

$\Theta(\mathbf{y})$, a linear cumulative map to get the *cumulative ordering map* $\bar{\Theta} = (\bar{\theta}_1, \bar{\theta}_2, \dots, \bar{\theta}_m)$ defined as

$$\bar{\theta}_i(\mathbf{y}) = \sum_{j=1}^i \theta_j(\mathbf{y}) \quad \text{for } i = 1, 2, \dots, m. \quad (19)$$

The coefficients of vector $\bar{\Theta}(\mathbf{y})$ express, respectively: the largest outcome, the total of the two largest outcomes, the total of the three largest outcomes, etc.

Directly from the definition of the map $\bar{\Theta}$, it follows that for any two achievement vectors $\mathbf{y}', \mathbf{y}'' \in Y$ equation $\bar{\Theta}(\mathbf{y}') = \bar{\Theta}(\mathbf{y}'')$ holds, if and only if \mathbf{y}' and \mathbf{y}'' have the same distribution of outcomes (i.e., $\Theta(\mathbf{y}') = \Theta(\mathbf{y}'')$). Similarly, inequality $\Theta(\mathbf{y}') \leq \Theta(\mathbf{y}'')$ implies $\bar{\Theta}(\mathbf{y}') \leq \bar{\Theta}(\mathbf{y}'')$ but the reverse implication is not valid. For instance, $\bar{\Theta}(2, 2, 2) = (2, 4, 6) \leq (3, 5, 6) = \bar{\Theta}(3, 2, 1)$ and simultaneously $\Theta(2, 2, 2) \not\leq \Theta(3, 2, 1)$.

The relation $\bar{\Theta}(\mathbf{y}') \leq \bar{\Theta}(\mathbf{y}'')$ was extensively analyzed within the theory of majorization (Marshall and Olkin, 1979), where it is called the relation of weak submajorization. The theory of majorization includes the results which allow us to derive the following theorem (Kostreva and Ogryczak, 1999a).

Theorem 1. *Achievement vector $\mathbf{y}' \in Y$ equitably dominates $\mathbf{y}'' \in Y$, if and only if $\bar{\theta}_i(\mathbf{y}') \leq \bar{\theta}_i(\mathbf{y}'')$ for all $i \in I$ where at least one strict inequality holds.*

In income economics the Lorenz curve is a popular tool to explain inequalities (Young, 1994). In the context of income distribution, the Lorenz curve is a cumulative population versus income curve. First, all individuals are ranked by income, from poorest to richest. For each rank, we compute the proportion of the income earned by all individuals at this rank and all ranks below this rank. The relationship between the proportions of population and income defines the Lorenz curve. A perfectly equal distribution of income has the diagonal line as the Lorenz curve. All other distributions generate convex Lorenz curves below the diagonal line.

Note that the definition of values $\bar{\theta}_i(\mathbf{y})$ for $i = 1, 2, \dots, m$ is similar to the construction of the Lorenz curve for the population of m clients

(outcomes). The main difference depends on inverse ordering, from the largest to the smallest value. It is due to minimization problem (8) opposite to the incomes. If considered in connection with some obnoxious quantity, we get the upper Lorenz curves which are concave and fall above the diagonal equity line. If the curve corresponding to distribution A falls below the curve corresponding to distribution B, then distribution A is considered as less unequal than the latter one.

Vector $\bar{\Theta}(\mathbf{y})$ can be viewed graphically with the Lorenz-type curve connecting point $(0, 0)$ and points $(i/m, \bar{\theta}_i(\mathbf{y})/m)$ for $i = 1, 2, \dots, m$. In the case of two achievement vectors $\mathbf{y}', \mathbf{y}'' \in Y$ with the same total of outcomes ($\bar{\theta}_m(\mathbf{y}') = \bar{\theta}_m(\mathbf{y}'')$), the inequality $\bar{\Theta}(\mathbf{y}') \leq \bar{\Theta}(\mathbf{y}'')$ is equivalent to the dominance \mathbf{y}' over \mathbf{y}'' in the sense of upper Lorenz curves. In the general case, the upper Lorenz curves may be considered the graphs of vectors $\bar{\Theta}(\mathbf{y})/\bar{\theta}_m(\mathbf{y})$. Graphs of vectors $\bar{\Theta}(\mathbf{y})$ take the form of unnormalized concave curves (Fig. 1), the *upper absolute Lorenz curves*. Note that in terms of the Lorenz curves no achievement vector can be better than the vector of equal outcomes. Equitable dominance (and the absolute Lorenz curves) takes into account also values of outcomes. Vectors of equal outcomes are distinguished according to the value of outcomes. They are graphically represented with various ascent lines in Fig. 1. With the relation of equitable dominance an achievement vector of small unequal $\bar{\theta}_m$ outcomes may be preferred to an achievement vector with large equal outcomes.

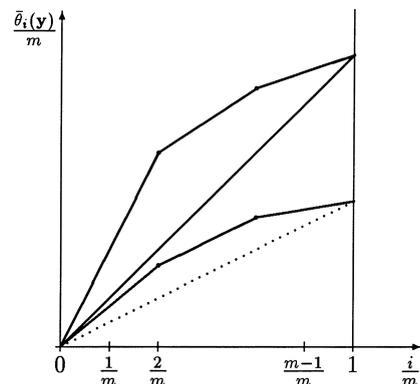


Fig. 1. $\bar{\Theta}(\mathbf{y})$ as upper absolute Lorenz curves.

Example 2. In order to illustrate the concept of equitable dominance, let us consider an example (Ogryczak, 1997) of location two facilities among 10 spatial units, where each spatial unit can be considered as a potential location. We assume that the facilities have unlimited capacities and each spatial unit represents one client ($v_i = 1$) to be served by the nearest facility. Thus, the problem takes the form (9)–(13) from Example 1 with $m = n = 10$ and $p = 2$. To make possible an easy analysis of the problem without complex computations, we consider several units U1, U2, . . . , U10 as points on a line, say the X-axis, with coordinates: 0, 4, 5, 6, 8, 17, 18, 19, 20 and 28, respectively.

Table 1 contains (four) various solutions to the location problem. The first one corresponds to the lexicographic center solution (Ogryczak, 1997), where in addition to the largest distance we minimize also the second largest distance, the third largest and so on. This solution depends on location facilities in spatial units U2 and U9. In the second row of Table 1 there are presented distances for another, in our opinion the worst, center solution. It is based on location facilities in spatial units U1 and U9. Further, we have included the median solution and the solution minimizing the Gini coefficient (7). The median solution is based on locations in units U3 and U8, whereas the Gini solution uses locations U1 and U10. Note that among four solutions (achievement vectors) pre-

sented in Table 1 no one is dominated by any other. In fact, all these solutions are efficient as, due to the problem specificity, each feasible solution is efficient.

Comparing cumulative ordered outcomes $\bar{\Theta}(\mathbf{y})$ given in Table 2, one can see that cumulative ordered achievement vector of the second solution is dominated by that of the first one. The cumulative ordered achievement vector of the fourth solution is dominated by each of other three vectors. Thus, both the second and the fourth solutions are not equitably efficient.

Note that Theorem 1 permits one to express the relationship between equitable efficiency for problem (8) and the Pareto-optimality for the multiple criteria problem with objectives $\bar{\Theta}(\mathbf{f}(\mathbf{x}))$:

$$\min\{(\bar{\theta}_1(\mathbf{f}(\mathbf{x})), \bar{\theta}_2(\mathbf{f}(\mathbf{x})), \dots, \bar{\theta}_m(\mathbf{f}(\mathbf{x}))) : \mathbf{x} \in Q\}. \tag{20}$$

Corollary 1. *A location pattern $\mathbf{x} \in Q$ is an equitably efficient solution of the multiple criteria problem (8), if and only if it is an efficient solution of the multiple criteria problem (20).*

Corollary 1 allows one to generate equitably efficient solutions of problem (8) as efficient solutions of problem (20) (cf. Kostreva and Ogryczak, 1999). The center solution concept (1)

Table 1
Outcomes of location solutions in Example 1

Solution		Outcomes y_i									
U2	U9	4	0	1	2	4	3	2	1	0	8
U1	U9	0	4	5	6	8	3	2	1	0	8
U3	U8	5	1	0	1	3	2	1	0	1	9
U1	U10	0	4	5	6	8	11	10	9	8	0

Table 2
Cumulative ordered outcomes in Example 1

Solution		Cumulative ordered outcomes $\bar{\theta}_i(\mathbf{y})$									
U2	U9	8	12	16	19	21	23	24	25	25	25
U1	U9	8	16	22	27	31	34	36	37	37	37
U3	U8	9	14	17	19	20	21	22	23	23	23
U1	U10	11	21	30	38	46	52	57	61	61	61

corresponds to minimization of the first objective in problem (20). Similarly, the median solution concept (2), minimizing the mean outcome, corresponds to minimization of the last (m th) objective in problem (20). Thus, both the concepts use only one objective in the multiple criteria problem (20). This is enough to guarantee that the unique center and the unique median are equitably efficient solutions. However, in the general case of multiple optimal solutions of the corresponding problem (1) or (2), respectively, some of center or median solutions may be equitably dominated. In fact, neither center nor median solution concept complies with the principle of transfers.

In the case of efficiency one may use the weighted sum of objective functions to generate various efficient solutions (Steuer, 1986). In the case of equitable multiple criteria programming one cannot assign various weights to individual objective functions, as that violates the requirement of impartiality (17). However, due to Corollary 1, the weighting approach can be applied to problem (20) resulting in the scalarization

$$\min \left\{ \sum_{i=1}^m w_i \bar{\theta}_i(\mathbf{f}(\mathbf{x})) : \mathbf{x} \in Q \right\}. \tag{21}$$

Note that, due to the definition of map $\bar{\theta}$ with Eq. (19), the above problem can be expressed in the form with weights $\bar{w}_i = \sum_{j=i}^m w_j$ ($i = 1, 2, \dots, m$) allocated to coefficients of the ordered achievement vector. Such an approach to multiple criteria optimization was introduced by Yager (1988) as the so-called *ordered weighted averaging (OWA)*. When applying OWA to problem (8) we get

$$\min \left\{ \sum_{i=1}^m w_i \theta_i(\mathbf{f}(\mathbf{x})) : \mathbf{x} \in Q \right\}. \tag{22}$$

If weights w_i are strictly decreasing and positive, i.e.,

$$w_1 > w_2 > \dots > w_{m-1} > w_m > 0, \tag{23}$$

then each optimal solution of the OWA problem (22) is an equitably efficient solution of location problem (8). Thus, the OWA approach defines a

parametric family of equitably efficient solution concepts for location problem (8).

As the limiting case of the OWA problem (22), when the differences among weights w_i tend to infinity, we get the lexicographic problem

$$\text{lex min}\{(\theta_1(\mathbf{f}(\mathbf{x})), \theta_2(\mathbf{f}(\mathbf{x})), \dots, \theta_m(\mathbf{f}(\mathbf{x}))) : \mathbf{x} \in Q\}, \tag{24}$$

where first $\theta_1(\mathbf{f}(\mathbf{x}))$ is minimized, next $\theta_2(\mathbf{f}(\mathbf{x}))$ and so on. Problem (24) represents the lexicographic minimax approach to the original multiple criteria problem (8). In the location context this solution concept is called the *lexicographic center* (Ogryczak, 1997). The lexicographic center is indeed a refinement (regularization) of the center solution concept (1), but in the former, in addition to the largest outcome, we minimize also the second largest outcome (provided that the largest one remains as small as possible), minimize the third largest (provided that the two largest remain as small as possible), and so on. Due to Eq. (19), problem (24) is equivalent to the problem

$$\text{lex min}\{(\bar{\theta}_1(\mathbf{f}(\mathbf{x})), \bar{\theta}_2(\mathbf{f}(\mathbf{x})), \dots, \bar{\theta}_m(\mathbf{f}(\mathbf{x}))) : \mathbf{x} \in Q\},$$

which can be considered the standard lexicographic optimization applied to problem (20). As the lexicographic optimization generates efficient solutions, thus due to Corollary 1, the optimal solution of the lexicographic minimax problem (24) is an equitably efficient solution of the multiple criteria problem (8).

Similarly, as the limiting case of the OWA problem (22), when the differences among weights w_i tend to zero, we get the lexicographic problem

$$\text{lex min}\{(\bar{\theta}_m(\mathbf{f}(\mathbf{x})), \bar{\theta}_{m-1}(\mathbf{f}(\mathbf{x})), \dots, \bar{\theta}_1(\mathbf{f}(\mathbf{x}))) : \mathbf{x} \in Q\}, \tag{25}$$

where first $\bar{\theta}_m(\mathbf{f}(\mathbf{x}))$ is minimized, next $\bar{\theta}_{m-1}(\mathbf{f}(\mathbf{x}))$ and so on. The problem (25) defines the solution concept of *lexicographic median* which is an equitably efficient refinement of the median (2).

The OWA model (22) defines the multidimensional continuum of equitably efficient location concepts spanning the space between the (lexicographic) center and the (lexicographic) median. Although rich with equitably efficient solutions,

the OWA approach, in general, is very hard to implement even for a relatively small problem size (small number of clients m). In addition, the OWA approach requires the disaggregation of location problem with the client weights v_i which usually dramatically increases the problem size.

4. Mean/equity trade-offs

The OWA model (22) defines the entire gamut of equitably efficient solutions to location problem (8), but it is hard to implement. As a simplified approach one may consider a bicriteria mean/equity model (Mandell, 1991):

$$\min\{\mu(\mathbf{f}(\mathbf{x})), \varrho(\mathbf{f}(\mathbf{x}))\} : \mathbf{x} \in Q \tag{26}$$

taking into account both the efficiency with minimization of the mean outcome $\mu(\mathbf{y})$ and the equity with minimization of an inequality measure $\varrho(\mathbf{y})$. For typical inequality measures bicriteria model (26) is computationally very attractive since both the criteria are well defined directly for the weighted location problem without necessity of its disaggregation. Moreover, $\mu(\mathbf{y})$ is a linear function of outcomes and absolute inequalities measures that we consider are convex piecewise linear functions which can be introduced into problem (26) with auxiliary linear inequalities. Exactly, the maximum (upside) deviation (6) can be implemented within problem (26) with a nonnegative variable Δ and the following linear inequalities:

$$\Delta \geq y_i - \sum_{j=1}^m \bar{v}_j y_j \quad \text{for } i = 1, 2, \dots, m.$$

The mean (absolute) deviation (5) can be implemented with auxiliary nonnegative variables z_i ($i = 1, 2, \dots, m$) and linear constraints:

$$\delta = \sum_{i=1}^m \bar{v}_i z_i,$$

$$z_i \geq y_i - \sum_{j=1}^m \bar{v}_j y_j \quad \text{for } i = 1, 2, \dots, m.$$

Similarly, the mean (absolute) difference (3) can be implemented with auxiliary nonnegative vari-

ables z_{ij} ($i = 1, 2, \dots, m; j = 1, 2, \dots, m$) and linear constraints:

$$D = \frac{1}{2} \sum_{i=1}^m \sum_{j=1}^m \bar{v}_i \bar{v}_j z_{ij}, \tag{27}$$

$$z_{ij} \geq y_i - y_j \quad \text{for } i = 1, 2, \dots, m, \quad j = 1, 2, \dots, m. \tag{28}$$

One may raise a question whether such bicriteria mean/equity models comply with the preference structure of the equitable efficient solution concept. In the case of bicriteria mean/equity problem (26) we get the following dominance relation:

$$\mathbf{y}' \preceq_{\mu/\varrho} \mathbf{y}'' \iff \mu(\mathbf{y}') \leq \mu(\mathbf{y}'') \text{ and } \varrho(\mathbf{y}') \leq \varrho(\mathbf{y}'').$$

One can easily notice that such a defined preference relation does not satisfy the requirement of the strict monotonicity (16) if function ϱ does not satisfy the condition of at least weak monotonicity, i.e., $\varrho(\mathbf{y} - \varepsilon \mathbf{e}_i) \leq \varrho(\mathbf{y})$ for $\varepsilon > 0$. Unfortunately, the property of weak monotonicity does not hold for the Gini coefficient (7) used by Mandell (1991), nor the other typical inequality measures. Thus, the mean/equity bicriteria approaches, in general, may generate solutions which are not equitably efficient. For instance, the last solution in Example 2 is the unique optimal solution for minimization of the Gini coefficient and thereby it is an efficient solution of the corresponding mean/equity problem whereas as shown in Table 2, it is equitably dominated by each of three other solutions.

An important advantage of mean/equity approaches is the possibility of a trade-off analysis. Having assumed a trade-off coefficient λ between the inequality measure $\varrho(\mathbf{y})$ and the mean outcome, one may directly compare real values of $\mu(\mathbf{y}) + \lambda\varrho(\mathbf{y})$ and find the best location pattern by solving the optimization problem

$$\min\{\mu(\mathbf{f}(\mathbf{x})) + \lambda\varrho(\mathbf{f}(\mathbf{x}))\} : \mathbf{x} \in Q. \tag{29}$$

The optimal solution of problem (29) we refer to as the λ -mean-equity solution. Solving parametric problem (29) with changing $\lambda > 0$ allows one to select an appropriate value of the trade-off coefficient λ and the corresponding optimal location pattern through a graphical analysis in the mean/

equity image space. In this section we show that the optimization (29) with trade-off coefficients bounded by 1 is consistent with equitable efficiency in the case of absolute inequalities measures that we consider.

Note that the maximum (upperside) deviation (6) can be expressed in terms of $\bar{\theta}_i(\mathbf{y})$ as follows:

$$\Delta(\mathbf{y}) = \max_{i=1,\dots,m} y_i - \frac{1}{m} \sum_{i=1}^m y_i = \bar{\theta}_1(\mathbf{y}) - \frac{1}{m} \bar{\theta}_m(\mathbf{y}). \quad (30)$$

It leads to the following result.

Theorem 2. *Except for location patterns with identical mean and maximum deviation, every location pattern $\mathbf{x} \in Q$ that is minimal for $\mu(\mathbf{f}(\mathbf{x})) + \lambda\Delta(\mathbf{f}(\mathbf{x}))$ with $0 < \lambda < 1$ is an equitably efficient solution of the location problem (8).*

Proof. Let $0 < \lambda \leq 1$ and $\mathbf{x}^0 \in Q$ be minimal by $\mu(\mathbf{f}(\mathbf{x})) + \lambda\Delta(\mathbf{f}(\mathbf{x}))$. Note that, due to Eq. (30), one gets

$$\begin{aligned} \mu(\mathbf{f}(\mathbf{x})) + \lambda\Delta(\mathbf{f}(\mathbf{x})) \\ = \lambda\bar{\theta}_1(\mathbf{f}(\mathbf{x})) + \frac{1-\lambda}{m} \bar{\theta}_m(\mathbf{f}(\mathbf{x})). \end{aligned} \quad (31)$$

Hence, in the case of $0 < \lambda < 1$, function $\mu(\mathbf{f}(\mathbf{x})) + \lambda\Delta(\mathbf{f}(\mathbf{x}))$ is a linear combination with positive coefficients (weights) of the objective functions $\bar{\theta}_1(\mathbf{f}(\mathbf{x}))$ and $\bar{\theta}_m(\mathbf{f}(\mathbf{x}))$.

Suppose there exists a location pattern $\mathbf{x}' \in Q$ which equitably dominates \mathbf{x}^0 . Then $\bar{\theta}_i(\mathbf{f}(\mathbf{x}')) \leq \bar{\theta}_i(\mathbf{f}(\mathbf{x}^0))$ and, in particular, $\bar{\theta}_1(\mathbf{f}(\mathbf{x}')) \leq \bar{\theta}_1(\mathbf{f}(\mathbf{x}^0))$ and $\bar{\theta}_m(\mathbf{f}(\mathbf{x}')) \leq \bar{\theta}_m(\mathbf{f}(\mathbf{x}^0))$. This together with a fact that \mathbf{x}^0 is optimal implies

$$\begin{aligned} \lambda\bar{\theta}_1(\mathbf{f}(\mathbf{x}')) + \frac{1-\lambda}{m} \bar{\theta}_m(\mathbf{f}(\mathbf{x}')) \\ = \lambda\bar{\theta}_1(\mathbf{f}(\mathbf{x}^0)) + \frac{1-\lambda}{m} \bar{\theta}_m(\mathbf{f}(\mathbf{x}^0)) \end{aligned}$$

and further $\bar{\theta}_1(\mathbf{f}(\mathbf{x}')) = \bar{\theta}_1(\mathbf{f}(\mathbf{x}^0))$ and $\bar{\theta}_m(\mathbf{f}(\mathbf{x}')) = \bar{\theta}_m(\mathbf{f}(\mathbf{x}^0))$. Hence, $\mu(\mathbf{f}(\mathbf{x}')) = \mu(\mathbf{f}(\mathbf{x}^0))$ and, from Eq. (30), $\Delta(\mathbf{f}(\mathbf{x}')) = \Delta(\mathbf{f}(\mathbf{x}^0))$ which completes the proof. \square

Theorem 2 partially justifies the maximum deviation λ -mean-equity as an equitably efficient solution concept. Note that, due to Eq. (31), its objective function can be expressed as follows:

$$\begin{aligned} \mu(\mathbf{f}(\mathbf{x})) + \lambda\Delta(\mathbf{f}(\mathbf{x})) \\ = \lambda \max_{i=1,\dots,m} f_i(\mathbf{x}) + (1-\lambda) \sum_{i=1}^m \bar{v}_i f_i(\mathbf{x}), \end{aligned} \quad (32)$$

which was introduced by Halpern (1978) to define his *convex λ -cent-dian* solution concept. Thus, Theorem 2 justifies also the convex λ -cent-dians as equitably efficient solutions to location problem (8).

The maximum deviation is an inequality measure related to the worst case analysis. It is in some manner very “rough” as it does not take into account the distribution of outcomes other than the worst one, which means that only two objective functions $\bar{\theta}_i(\mathbf{y})$ from the multiple criteria problem (20) are used. Similar results can be established for absolute inequality measures taking into account all the quantities $\bar{\theta}_i(\mathbf{y})$.

The mean deviation (5) can be expressed in terms of $\bar{\theta}_i(\mathbf{y})$ as follows:

$$\begin{aligned} \delta(\mathbf{y}) &= \frac{1}{m} \sum_{i:\bar{\theta}_i(\mathbf{y}) > \mu(\mathbf{y})} [\bar{\theta}_i(\mathbf{y}) - \mu(\mathbf{y})] \\ &= \frac{1}{m} \max_{i=1,\dots,m-1} \left[\bar{\theta}_i(\mathbf{y}) - \frac{i}{m} \bar{\theta}_m(\mathbf{y}) \right]. \end{aligned} \quad (33)$$

It leads to the following result.

Theorem 3. *Except for location patterns with identical mean and mean deviation, every location pattern $\mathbf{x} \in Q$ that is minimal for $\mu(\mathbf{f}(\mathbf{x})) + \lambda\delta(\mathbf{f}(\mathbf{x}))$ with $0 < \lambda \leq 1$ is an equitably efficient solution of the location problem (8).*

Proof. Let $0 < \lambda \leq 1$ and $\mathbf{x}^0 \in Q$ be minimal by $\mu(\mathbf{f}(\mathbf{x})) + \lambda\delta(\mathbf{f}(\mathbf{x}))$. Note that, due to Eq. (33), one gets

$$\begin{aligned} \mu(\mathbf{f}(\mathbf{x})) + \lambda\delta(\mathbf{f}(\mathbf{x})) \\ = \frac{1}{m} \bar{\theta}_m(\mathbf{f}(\mathbf{x})) + \frac{\lambda}{m} \max_{i=1,\dots,m-1} \left[\bar{\theta}_i(\mathbf{f}(\mathbf{x})) - \frac{i}{m} \bar{\theta}_m(\mathbf{f}(\mathbf{x})) \right] \\ = \max_{i=1,\dots,m-1} \left[\frac{\lambda}{m} \bar{\theta}_i(\mathbf{f}(\mathbf{x})) + \frac{m-i\lambda}{m^2} \bar{\theta}_m(\mathbf{f}(\mathbf{x})) \right]. \end{aligned} \quad (34)$$

Thus, \mathbf{x}^0 is an optimal solution to the minimax scalarization of the multiple criteria problem:

$$\min\{g_1(\mathbf{f}(\mathbf{x})), g_2(\mathbf{f}(\mathbf{x})), \dots, g_{m-1}(\mathbf{f}(\mathbf{x}))\}: \mathbf{x} \in Q\}, \tag{35}$$

with $m - 1$ objective functions g_i given by the formula

$$g_i(\mathbf{y}) = \frac{\lambda}{m} \bar{\theta}_i(\mathbf{y}) + \frac{m - i\lambda}{m^2} \bar{\theta}_m(\mathbf{y})$$

for $i = 1, 2, \dots, m - 1$. (36)

Moreover, both the coefficients in Eq. (36) are positive and therefore every efficient solution of problem (35) is also an efficient solution of problem (20).

Suppose there exists a location pattern $\mathbf{x}' \in Q$ which equitably dominates \mathbf{x}^0 . Then $\bar{\Theta}(\mathbf{f}(\mathbf{x}')) \leq \bar{\Theta}(\mathbf{f}(\mathbf{x}^0))$ and, due to positive coefficients in Eq. (36), $g_i(\mathbf{f}(\mathbf{x}')) \leq g_i(\mathbf{f}(\mathbf{x}^0))$ for $i = 1, 2, \dots, m - 1$. On the other hand, $\max_{i=1, \dots, m-1} g_i(\mathbf{f}(\mathbf{x}')) \geq \max_{i=1, \dots, m-1} g_i(\mathbf{f}(\mathbf{x}^0))$. Hence, there exists index i_0 such that $g_{i_0}(\mathbf{f}(\mathbf{x}')) = g_{i_0}(\mathbf{f}(\mathbf{x}^0))$ and therefore $\bar{\theta}_m(\mathbf{f}(\mathbf{x}')) = \bar{\theta}_m(\mathbf{f}(\mathbf{x}^0))$. Thus, $\mu(\mathbf{f}(\mathbf{x}')) = \mu(\mathbf{f}(\mathbf{x}^0))$ and $\delta(\mathbf{f}(\mathbf{x}')) = \delta(\mathbf{f}(\mathbf{x}^0))$ which completes the proof. \square

The mean difference (3) can be expressed in terms of $\bar{\theta}_i(\mathbf{y})$ as follows:

$$D(\mathbf{y}) = \frac{1}{m^2} \sum_{i=1}^{m-1} [(i + 1)\bar{\theta}_i(\mathbf{y}) - i\bar{\theta}_{i+1}(\mathbf{y})]$$

$$= \frac{2}{m^2} \sum_{i=1}^{m-1} \bar{\theta}_i(\mathbf{y}) - \frac{m - 1}{m^2} \bar{\theta}_m(\mathbf{y}). \tag{37}$$

It leads to the following result.

Theorem 4. *Every location pattern $\mathbf{x} \in Q$ that is minimal for $\mu(\mathbf{f}(\mathbf{x})) + \lambda D(\mathbf{f}(\mathbf{x}))$ with $0 < \lambda \leq 1$ is an equitably efficient solution of the location problem (8).*

Proof. Let $0 < \lambda \leq 1$ and $\mathbf{x}^0 \in Q$ be minimal by $\mu(\mathbf{f}(\mathbf{x})) + \lambda D(\mathbf{f}(\mathbf{x}))$. Note that, due to Eq. (37), one gets

$$\mu(\mathbf{y}) + \lambda D(\mathbf{y}) = \frac{2\lambda}{m^2} \sum_{i=1}^{m-1} \bar{\theta}_i(\mathbf{y}) + \frac{m - \lambda(m - 1)}{m^2} \bar{\theta}_m(\mathbf{y}). \tag{38}$$

Hence, in the case of $0 < \lambda \leq 1$, function $\mu(\mathbf{f}(\mathbf{x})) + \lambda D(\mathbf{f}(\mathbf{x}))$ is a linear combination with positive weights of the objective functions $\bar{\theta}_i(\mathbf{f}(\mathbf{x}))$ for $i = 1, 2, \dots, m$. Therefore, \mathbf{x}^0 is an efficient solution of the multiple criteria problem (20) and, due to Corollary 1, \mathbf{x}^0 is an equitably efficient solution of the location problem (8). \square

We have shown that the λ -mean-equity solution concepts corresponding to all three inequality measures, in the case of $0 < \lambda < 1$, can be considered scalarizations of the multiple criteria problem (20). This can be illustrated in the Lorenz-type diagram, we considered in the previous section (Fig. 1). Recall that, under assumption of positive outcomes, vector $\bar{\Theta}(\mathbf{y})$ can be then viewed graphically with the upper absolute Lorenz curve connecting point $(0, 0)$ and points $(i/m, \bar{\theta}_i(\mathbf{y})/m)$ for $i = 1, 2, \dots, m$ where the last point (for $i = m$) is $(1, \mu(\mathbf{y}))$. Note that in our model the perfectly equal achievement vector with mean value $\mu(\mathbf{y})$ has all the coefficients equal to $\mu(\mathbf{y})$ and its absolute Lorenz curve is the ascent line connecting points $(0, 0)$ and $(1, \mu(\mathbf{y}))$. Hence, the space between the curve $(i/m, \bar{\theta}_i(\mathbf{y})/m)$ and its ascent line represents the dispersion (and thereby the inequality) of \mathbf{y} in comparison to the perfectly equal achievement vector of $\mu(\mathbf{y})$. We shall call it the *dispersion space*. Both size and shape of the dispersion space are important for complete description of the inequality. Nevertheless, it is quite natural to consider some size parameters as summary characteristics of the inequality. As shown in Fig. 2, all three inequality measures, we have considered, represent some size parameters of the dispersion space. Note that vertical diameter of the dispersion space at point i/m is given as $\delta_i(\mathbf{y}) = (1/m)\bar{\theta}_i(\mathbf{y}) - (i/m^2)\bar{\theta}_m(\mathbf{y})$. Hence, for the mean deviation, due to Eq. (33), we get $\delta(\mathbf{y}) = \max_{i=1, \dots, m} \delta_i(\mathbf{y})$. This means that $\delta(\mathbf{y})$ represents the largest vertical diameter of the dispersion space. Similarly, for the maximum deviation, due to Eq. (30), we get $\Delta(\mathbf{y}) = m\delta_1(\mathbf{y})$. Thus, $\Delta(\mathbf{y})$

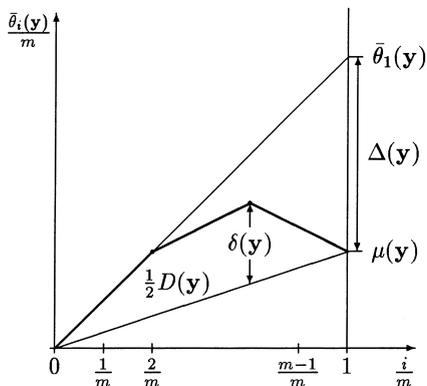


Fig. 2. $\bar{\Theta}(\mathbf{y})$ and inequality measures.

represents the projection of $\delta_1(\mathbf{y})$ onto the vertical line at $i = m$ or the largest vertical diameter of the corresponding triangular envelope of the dispersion space. The mean difference, due to Eq. (37), satisfies $D(\mathbf{y}) = (2/m) \sum_{i=1}^{m-1} \delta_i(\mathbf{y})$. That means, $D(\mathbf{y})$ is twice the area of the dispersion space. This explains why for this inequality measure we get the strongest result (Theorem 4) in the sense that each λ -mean-equity solution with bounded trade-off λ is an equitably efficient solution of the location problem (8). On the other hand, the mean difference is computationally more complex than other two measures. Note that implementation of the mean difference requires m^2 auxiliary inequalities (28) whereas the other two measures need only m auxiliary inequalities.

The λ -mean-equity solution concept with mean difference $D(\mathbf{y})$ as the inequality measure, due to Eq. (38), may be viewed as the OWA aggregation (22) with weights $w_i = (m + (m - 2i + 1)\lambda)/m^2$ for $i = 1, 2, \dots, m$. Hence, for the trade-off coefficient $0 < \lambda \leq 1$ the weights are positive and strictly decreasing (23) which causes that every optimal solution is equitably efficient. However, $w_i - w_{i+1} = 2\lambda/m^2$ for all $i = 1, 2, \dots, m - 1$. Thus, this

approach, in terms of the OWA aggregation, considers only weights decreasing by a constant step.

5. Other bicriteria approaches

In the case when the multiple criteria problem (8) is a discrete one (like the discrete location problem (9)–(13) in Example 1), there exist equitably efficient location patterns that are efficient solutions of the bicriteria mean/equity problem (26), but they cannot be generated as λ -mean-equity solutions. We illustrate this with a small example.

Example 3. Let us consider a simple single facility location problem with two clients (C1 and C2) and three potential locations (P1, P2 and P3). The distances between the clients and potential locations are given as follows: $d_{11} = 10, d_{12} = 12.8, d_{13} = 15, d_{21} = 17, d_{22} = 16, d_{23} = 15$.

The problem has, obviously, three feasible solutions corresponding to potential locations P1, P2 and P3, respectively. Quantitative characteristics of these solutions are given in Table 3. Note that all three feasible solutions are equitably efficient and they are efficient solutions of bicriteria mean/equity problem (26) for all considered absolute inequality measures. Nevertheless, one can easily verify that while dealing with the trade-off analysis (29), location P2 cannot be selected for any positive trade-off coefficient λ . Location P2 is never a λ -mean-equity solution for the data of this example. Moreover, location P2 is never an optimal solution solutions to the OWA problem (22) with positive weights.

In the case of discrete location problems not all efficient solutions of the mean/equity problem (26)

Table 3
Locations in Example 3

	\mathbf{y}		$\bar{\Theta}(\mathbf{y})$	$\mu(\mathbf{y})$	$\Delta(\mathbf{y})$	$\delta(\mathbf{y})$	$D(\mathbf{y})$
P1	10.0	17.0	17.0	27.0	13.5	3.5	1.75
P2	12.8	16.0	16.0	28.8	14.4	1.6	0.80
P3	15.0	15.0	15.0	30.0	15.0	0.0	0.00

can be identified as λ -mean-equity solutions. This is a direct consequence of the well-known flaws of convex weighting approaches to nonconvex multiple criteria optimization (Steuer, 1986). Therefore, similar difficulties one may encounter (Ogryczak, 1997a) while dealing with location problems on general networks (Labbé et al., 1996). Unfortunately, one cannot simply apply another multiple criteria approach to the bicriteria mean/equity problem (26) since not all its efficient solutions are equitably efficient solutions of the location problem (8). This difficulty can be overcome by replacing the original mean/equity bicriteria optimization (26) with the following bicriteria problem:

$$\min\{(\mu(\mathbf{f}(\mathbf{x})), M(\mathbf{f}(\mathbf{x}))) : \mathbf{x} \in Q\}, \tag{39}$$

where the second objective represents the sum of the mean outcome and the corresponding inequality measure, i.e.,

$$M(\mathbf{y}) = \mu(\mathbf{y}) + \varrho(\mathbf{y}). \tag{40}$$

Note that, in the case of maximum deviation $\Delta(\mathbf{y})$, the corresponding quantity $M(\mathbf{y})$ represents the worst outcome:

$$M_{\Delta}(\mathbf{y}) = \mu(\mathbf{y}) + \Delta(\mathbf{y}) = \max_{i=1, \dots, m} y_i.$$

Similarly, in the case of mean deviation we get the mean worse-side outcome:

$$M_{\delta}(\mathbf{y}) = \mu(\mathbf{y}) + \delta(\mathbf{y}) = \sum_{i=1}^m \bar{v}_i \max\{y_i, \mu(\mathbf{y})\},$$

and, in the case of mean difference we get the mean pairwise worse outcome:

$$M_D(\mathbf{y}) = \mu(\mathbf{y}) + D(\mathbf{y}) = \sum_{i=1}^m \sum_{j=1}^m \bar{v}_i \bar{v}_j \max\{y_i, y_j\}.$$

Thus, $M(\mathbf{y})$ may be considered a measure related worst value and we will refer to the bicriteria problem (39) as to mean/worst problem.

For any inequality measure $M(\mathbf{y}) \geq \mu(\mathbf{y})$, since $\varrho(\mathbf{y}) \geq 0$. One can easily verify (cf. Fig. 2) that for the absolute inequality measures, we consider, the following inequality holds:

$$\mu(\mathbf{y}) \leq M_{\delta}(\mathbf{y}) \leq M_D(\mathbf{y}) \leq M_{\Delta}(\mathbf{y}).$$

While applying the standard weighting approach (Steuer, 1986) to the bicriteria mean/worst problem (39) one gets the parametric objective function:

$$\begin{aligned} H_{\lambda}(\mathbf{x}) &= (1 - \lambda)\mu(\mathbf{f}(\mathbf{x})) + \lambda M(\mathbf{f}(\mathbf{x})) \\ &= \mu(\mathbf{f}(\mathbf{x})) + \lambda \varrho(\mathbf{f}(\mathbf{x})), \quad 0 < \lambda < 1. \end{aligned} \tag{41}$$

Hence, all the λ -mean-equity solution concepts, we considered in the previous section, can be viewed as weighting approaches to the corresponding mean/worst problems (39), like the convex λ -centdians (32) in the case of equity measured with maximum deviation. It turns out, however, that all efficient solutions of the mean/worst problems has the same equitability properties as the corresponding λ -mean-equity solutions. This is shown in the following theorems.

Theorem 5. *Except for location patterns with identical mean $\mu(\mathbf{y})$ and worst outcome $M_{\Delta}(\mathbf{y})$, every efficient solution to the bicriteria problem*

$$\min\{(\mu(\mathbf{f}(\mathbf{x})), M_{\Delta}(\mathbf{f}(\mathbf{x}))) : \mathbf{x} \in Q\} \tag{42}$$

is an equitably efficient solution of the location problem (8).

Proof. Let $\mathbf{x}^0 \in Q$ be an efficient solution of problem (42). Suppose there exists a location pattern $\mathbf{x}' \in Q$ which equitably dominates \mathbf{x}^0 . Then $\bar{\Theta}(\mathbf{f}(\mathbf{x}')) \leq \bar{\Theta}(\mathbf{f}(\mathbf{x}^0))$ and, in particular, $\bar{\theta}_1(\mathbf{f}(\mathbf{x}')) \leq \bar{\theta}_1(\mathbf{f}(\mathbf{x}^0))$ and $\bar{\theta}_m(\mathbf{f}(\mathbf{x}')) \leq \bar{\theta}_m(\mathbf{f}(\mathbf{x}^0))$. Hence, $\mu(\mathbf{f}(\mathbf{x}')) \leq \mu(\mathbf{f}(\mathbf{x}^0))$ and $M_{\Delta}(\mathbf{f}(\mathbf{x}')) \leq M_{\Delta}(\mathbf{f}(\mathbf{x}^0))$. This together with the fact that \mathbf{x}^0 is efficient implies $\mu(\mathbf{f}(\mathbf{x}')) = \mu(\mathbf{f}(\mathbf{x}^0))$ and $M_{\Delta}(\mathbf{f}(\mathbf{x}')) = M_{\Delta}(\mathbf{f}(\mathbf{x}^0))$ which completes the proof. \square

Theorem 6. *Except for location patterns with identical mean $\mu(\mathbf{y})$ and mean worse-side outcome $M_{\delta}(\mathbf{y})$, every efficient solution to the bicriteria problem*

$$\min\{(\mu(\mathbf{f}(\mathbf{x})), M_{\delta}(\mathbf{f}(\mathbf{x}))) : \mathbf{x} \in Q\} \tag{43}$$

is an equitably efficient solution of the location problem (8).

Proof. Let $\mathbf{x}^0 \in Q$ be an efficient solution of problem (43). Suppose there exists a location pattern $\mathbf{x}' \in Q$ which equitably dominates \mathbf{x}^0 . Then $\bar{\Theta}(\mathbf{f}(\mathbf{x}')) \leq \bar{\Theta}(\mathbf{f}(\mathbf{x}^0))$. Hence, $\mu(\mathbf{f}(\mathbf{x}')) \leq \mu(\mathbf{f}(\mathbf{x}^0))$ and, due to (34) with $\lambda = 1$, $M_\delta(\mathbf{f}(\mathbf{x}')) \leq M_\delta(\mathbf{f}(\mathbf{x}^0))$. This together with a fact that \mathbf{x}^0 is efficient implies $\mu(\mathbf{f}(\mathbf{x}')) = \mu(\mathbf{f}(\mathbf{x}^0))$ and $M_\delta(\mathbf{f}(\mathbf{x}')) = M_\delta(\mathbf{f}(\mathbf{x}^0))$. \square

Theorem 7. Every efficient solution to the bicriteria problem

$$\min\{(\mu(\mathbf{f}(\mathbf{x})), M_D(\mathbf{f}(\mathbf{x}))) : \mathbf{x} \in Q\} \quad (44)$$

is an equitably efficient solution of the location problem (8).

Proof. Let $\mathbf{x}^0 \in Q$ be an efficient solution of problem (44). Suppose there exists a location pattern $\mathbf{x}' \in Q$ which equitably dominates \mathbf{x}^0 . Then $\bar{\theta}_i(\mathbf{f}(\mathbf{x}')) \leq \bar{\theta}_i(\mathbf{f}(\mathbf{x}^0))$ for $i = 1, 2, \dots, m$ where at least one inequality is strict. Hence, $\mu(\mathbf{f}(\mathbf{x}')) \leq \mu(\mathbf{f}(\mathbf{x}^0))$ and, due to (34) with $\lambda = 1$, $M_D(\mathbf{f}(\mathbf{x}')) < M_D(\mathbf{f}(\mathbf{x}^0))$ which contradicts to a fact that \mathbf{x}^0 is efficient. Thus, \mathbf{x}^0 is an equitably efficient solution of the location problem (8). \square

According to the theory of multiple criteria optimization (Steuer, 1986), in the case of non-convex problems, the whole set of efficient solutions can be completely parameterized with minimization of the weighted Chebyshev norm. Moreover, this optimization should be supported by some regularization (refinement) in the case of nonunique optimal solution. Let us define

$$\bar{H}_\lambda(\mathbf{x}) = \max\{(1 - \lambda)\mu(\mathbf{f}(\mathbf{x})), \lambda M(\mathbf{f}(\mathbf{x}))\}. \quad (45)$$

We call a location pattern $\mathbf{x} \in Q$ the *Chebyshev λ -mean-worst* solution if it is an optimal solution of the following lexicographic (two-level) problem:

$$\text{lex min}\{[\bar{H}_\lambda(\mathbf{x}), H_\lambda(\mathbf{x})] : \mathbf{x} \in Q\}. \quad (46)$$

The lexicographic minimization in problem (46) means that first we minimize $\bar{H}_\lambda(\mathbf{x})$ on $\mathbf{x} \in Q$, and next we minimize $H_\lambda(\mathbf{x})$ on the optimal set of $\bar{H}_\lambda(\mathbf{x})$. Thus, function $H_\lambda(\mathbf{x})$, defined as the convex linear combination (41), is used in problem (46) only for regularization purposes, in the case of nonunique minimum for the main function $\bar{H}_\lambda(\mathbf{x})$

defined with (45). However, this regularization is necessary to guarantee that the Chebyshev λ -mean-worst solutions are efficient solutions of the corresponding bicriteria mean/worst problem (39).

Theorem 8. For any location problem (8) and any absolute inequality measure q , a location pattern $\mathbf{x} \in Q$ is an efficient solution to the corresponding mean/worst problem (39), if and only if \mathbf{x} is the corresponding Chebyshev λ -mean-worst solution for some $0 < \lambda < 1$.

Proof. Let $\mathbf{x}^0 \in Q$ be a Chebyshev λ -mean-worst solution for some $0 < \lambda < 1$. Suppose that \mathbf{x}^0 is not efficient in problem (39). It means, there exists $\mathbf{x}' \in Q$ such that $M(\mathbf{f}(\mathbf{x}')) \leq M(\mathbf{f}(\mathbf{x}^0))$ and $\mu(\mathbf{f}(\mathbf{x}')) \leq \mu(\mathbf{f}(\mathbf{x}^0))$ where at least one inequality is strict. Hence, due to $0 < \lambda < 1$, we get $\bar{H}_\lambda(\mathbf{x}') \leq \bar{H}_\lambda(\mathbf{x}^0)$ and $H_\lambda(\mathbf{x}') < H_\lambda(\mathbf{x}^0)$ which contradicts optimality of \mathbf{x}^0 for problem (46). Thus, \mathbf{x}^0 is an efficient solution of the corresponding problem (39).

Let \mathbf{x}^0 be an efficient solution of the mean/worst problem (39). Recall that in our location problem all the outcomes (distances) are assumed to be nonnegative. If $\mu(\mathbf{f}(\mathbf{x}^0)) = M(\mathbf{f}(\mathbf{x}^0)) = 0$, then \mathbf{x}^0 is the unique Chebyshev λ -mean-worst solution for any $0 < \lambda < 1$. Otherwise $M(\mathbf{f}(\mathbf{x}^0)) > 0$. Let us define

$$\lambda = \mu(\mathbf{f}(\mathbf{x}^0)) / (M(\mathbf{f}(\mathbf{x}^0)) + \mu(\mathbf{f}(\mathbf{x}^0))).$$

Then $0 < \lambda < 1$,

$$1 - \lambda = M(\mathbf{f}(\mathbf{x}^0)) / (M(\mathbf{f}(\mathbf{x}^0)) + \mu(\mathbf{f}(\mathbf{x}^0)))$$

and

$$\begin{aligned} \bar{H}_\lambda(\mathbf{x}^0) &= \frac{M(\mathbf{f}(\mathbf{x}^0))\mu(\mathbf{f}(\mathbf{x}^0))}{M(\mathbf{f}(\mathbf{x}^0)) + \mu(\mathbf{f}(\mathbf{x}^0))} = \lambda M(\mathbf{f}(\mathbf{x}^0)) \\ &= (1 - \lambda)\mu(\mathbf{f}(\mathbf{x}^0)). \end{aligned} \quad (47)$$

Suppose that \mathbf{x}^0 is not the corresponding Chebyshev λ -mean-worst solution. Thus, there exists $\mathbf{x}' \in Q$ such that $\lambda M(\mathbf{f}(\mathbf{x}')) \leq \bar{H}_\lambda(\mathbf{x}^0)$ and $(1 - \lambda)\mu(\mathbf{f}(\mathbf{x}')) \leq \bar{H}_\lambda(\mathbf{x}^0)$ where at least one inequality is strict. Due to (47), it would contradict

efficiency of \mathbf{x}^0 . Thus, \mathbf{x}^0 must be the corresponding Chebyshev λ -mean-worst solution. \square

Theorem 8 together with earlier proven theorems for the specific inequality measures allows us to derive the following corollaries for the inequality measures, we consider.

Corollary 2. *Except for location patterns with identical mean $\mu(\mathbf{y})$ and maximum deviation $\Delta(\mathbf{y})$, any Chebyshev λ -mean-worst solution*

$$\begin{aligned} \text{lex min} \{ & [\max\{(1 - \lambda)\mu(\mathbf{f}(\mathbf{x})), \lambda M_\Delta(\mathbf{f}(\mathbf{x}))\}], \\ & (1 - \lambda)\mu(\mathbf{f}(\mathbf{x})) + \lambda M_\Delta(\mathbf{f}(\mathbf{x}))\} : \mathbf{x} \in Q\}, \\ & 0 < \lambda < 1 \end{aligned}$$

is an equitably efficient solution of the location problem (8).

Corollary 3. *Except for location patterns with identical mean $\mu(\mathbf{y})$ and mean deviation $\delta(\mathbf{y})$, any Chebyshev λ -mean-worst solution*

$$\begin{aligned} \text{lex min} \{ & [\max\{(1 - \lambda)\mu(\mathbf{f}(\mathbf{x})), \lambda M_\delta(\mathbf{f}(\mathbf{x}))\}], \\ & (1 - \lambda)\mu(\mathbf{f}(\mathbf{x})) + \lambda M_\delta(\mathbf{f}(\mathbf{x}))\} : \mathbf{x} \in Q\}, \\ & 0 < \lambda \leq 1 \end{aligned}$$

is an equitably efficient solution of the location problem (8).

Corollary 4. *Any Chebyshev λ -mean-worst solution*

$$\begin{aligned} \text{lex min} \{ & [\max\{(1 - \lambda)\mu(\mathbf{f}(\mathbf{x})), \lambda M_D(\mathbf{f}(\mathbf{x}))\}], \\ & (1 - \lambda)\mu(\mathbf{f}(\mathbf{x})) + \lambda M_D(\mathbf{f}(\mathbf{x}))\} : \mathbf{x} \in Q\}, \\ & 0 < \lambda \leq 1 \end{aligned}$$

is an equitably efficient solution of the location problem (8).

The Chebyshev λ -mean-worst approach, similar to the λ -mean-equity one, is a parametric solution concept generating various solutions depending on the value of the trade-off coefficient $0 < \lambda < 1$. While the λ -mean-equity solution concepts can be considered as generalization of the convex λ -cent-dians (Halpern, 1978) for other ab-

solute inequality measures, the Chebyshev λ -mean-worst solution concepts generalize the Chebyshev λ -cent-dian (Ogryczak, 1997a). According to Theorem 8, each efficient solution of the bicriteria mean/worst problem (39) can be found as a Chebyshev λ -mean-worst solution. Thus, the concept of Chebyshev λ -mean-worst solutions allows us to identify various equitably efficient solutions of the location problem (8) through modeling all rational compromises between the values of the mean and the (measure related) worst value. Note that equitably efficient location P2 from Example 3, which could never be identified as a λ -mean-equity solution, can be found as the Chebyshev λ -mean-worst solution.

Selection of trade-off coefficient λ depends on the type of a compromise one seeks. The Chebyshev λ -mean-worst solution is the median for λ close enough to 0, and the solution minimizing $M(\mathbf{f}(\mathbf{x}))$ (the center in the case of inequality measure $\Delta(\mathbf{y})$) for $\lambda \geq 1/2$ (since $M(\mathbf{y}) \geq \mu(\mathbf{y})$). For λ between 0 and 1/2 one may expect various compromise solutions. One may proceed the search for a satisfactory compromise in an interactive way. For more intuitive understanding of the trade-off λ , one may use the concept of Chebyshev λ -mean-worst solutions applied to the normalized objective functions

$$\bar{M}(\mathbf{f}(\mathbf{x})) = \frac{M(\mathbf{f}(\mathbf{x})) - M(\mathbf{f}(\mathbf{x}^M)) + \varepsilon}{M(\mathbf{f}(\mathbf{x}^\mu)) - M(\mathbf{f}(\mathbf{x}^M)) + \varepsilon}$$

and

$$\bar{\mu}(\mathbf{f}(\mathbf{x})) = \frac{\mu(\mathbf{f}(\mathbf{x})) - \mu(\mathbf{f}(\mathbf{x}^\mu)) + \varepsilon}{\mu(\mathbf{f}(\mathbf{x}^M)) - \mu(\mathbf{f}(\mathbf{x}^\mu)) + \varepsilon},$$

where

$$\mathbf{x}^M = \arg \min \{M(\mathbf{f}(\mathbf{x})) : \mathbf{x} \in Q\},$$

$$\mathbf{x}^\mu = \arg \min \{\mu(\mathbf{f}(\mathbf{x})) : \mathbf{x} \in Q\}$$

and ε is an arbitrarily small positive number introduced to guarantee positive values of the functions. Functions $\bar{M}(\mathbf{f}(\mathbf{x}))$ and $\bar{\mu}(\mathbf{f}(\mathbf{x}))$ represent the relative degradations of the corresponding functions $M(\mathbf{f}(\mathbf{x}))$ and $\mu(\mathbf{f}(\mathbf{x}))$ to their optimal values $M(\mathbf{x}^M)$ and $\mu(\mathbf{x}^\mu)$, respectively. One may

easily prove an analog of Theorem 8 and its corollaries for the Chebyshev λ -mean-worst solutions defined with the use of functions $\bar{M}(\mathbf{f}(\mathbf{x}))$ and $\bar{\mu}(\mathbf{f}(\mathbf{x}))$ instead of the original $M(\mathbf{f}(\mathbf{x}))$ and $\mu(\mathbf{f}(\mathbf{x}))$. Such a Chebyshev λ -mean-worst solution concept may be considered a special case of the reference point approach in multiple criteria optimization (cf. Wierzbicki, 1982).

6. Concluding remarks

While making location decisions, the distribution of travel distances among the service recipients (clients) is an important issue. It is usually tackled with the center or the median solution concepts. Both concepts minimize only simple scalar characteristics of the distribution: the maximal distance and the average distance, respectively. The entire distribution of distances can be taken into account in the multiple criteria model where all the distances (or more general effects) for the individual clients are considered as the set of uniform criteria to be minimized. In order to comply with the minimization of distances as well as with an equal consideration of the clients, the concept of equitable efficiency must be used for this multiple criteria model. The concept is based on extension of the standard efficiency concept with the principle of impartiality and the principle of transfers.

Equitably efficient solution concepts may be modeled with the standard multiple criteria optimization applied to the cumulative ordered outcomes. Although rich with equitably efficient solutions, these approaches, in general, are very hard to implement even for a relatively small problem size. In addition, the ordering of outcomes requires the disaggregation of location problem with the client weights which usually dramatically increases the problem size. Therefore, we are interested in solution concepts which can be applied directly to the weighted problem. As a simplified approach one may consider a bicriteria mean/equity model taking into account both the efficiency with minimization of the mean outcome and the equity with minimization of an inequality measure. For typical inequality measures such a

bicriteria model is computationally very attractive since both the criteria are well defined directly for the weighted location problem without necessity of its disaggregation. The mean/equity bicriteria approaches, in general, may generate solutions which are not equitably efficient. It turns out that, under the assumption of bounded trade-offs, the bicriteria mean/equity approaches for selected absolute inequality measures (maximum deviation, mean deviation or mean difference) comply with the rules of equitable multiple criteria optimization. Moreover, these absolute inequality measures can be used to build generalized bicriteria approaches not using directly the trade-off technique. Thus, the selected absolute inequality measures can be effectively used to incorporate equity factors into various facility location decision models.

This paper focuses on location problems. However, the location decisions are analyzed from the perspective of their effects for individual clients. Therefore, the general concept of the proposed equitable approaches can be used for optimization of various systems which serve many users. Moreover, uniform individual objectives may be associated with some events rather than the physical users, like in many dynamic optimization problems where uniform individual criteria represent the same outcome for various periods.

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