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Abstract	In this paper we analyze robust approaches to decision making under uncertainty where the expected outcome is maximized but the probabilities are known imprecisely. A conservative robust approach takes into account any probability distribution thus leading to the notion of robustness focusing on the worst case scenario and resulting in the max-min optimization. We consider softer robust models allowing the probabilities to vary only within given intervals. We show that the robust solution for only upper bounded probabilities becomes the tail mean, known also as the conditional value-at-risk (CVaR), with an appropriate tolerance level. For proportional upper and lower probability limits the corresponding robust solution may be expressed by the optimization of appropriately combined the mean and the tail mean criteria. Finally, a general robust solution for any arbitrary intervals of probabilities can be expressed with the optimization problem very similar to the tail mean and thereby easily implementable with auxiliary linear inequalities.

Robust Decisions Under Risk for Imprecise Probabilities

Włodzimierz Ogryczak

Abstract In this paper we analyze robust approaches to decision making under uncertainty where the expected outcome is maximized but the probabilities are known imprecisely. A conservative robust approach takes into account any probability distribution thus leading to the notion of robustness focusing on the worst case scenario and resulting in the max-min optimization. We consider softer robust models allowing the probabilities to vary only within given intervals. We show that the robust solution for only upper bounded probabilities becomes the tail mean, known also as the conditional value-at-risk (CVaR), with an appropriate tolerance level. For proportional upper and lower probability limits the corresponding robust solution may be expressed by the optimization of appropriately combined the mean and the tail mean criteria. Finally, a general robust solution for any arbitrary intervals of probabilities can be expressed with the optimization problem very similar to the tail mean and thereby easily implementable with auxiliary linear inequalities.

1 Introduction

Several approaches have been developed to deal with uncertain or imprecise data in optimization problems. In the standard stochastic programming models, we assume that the probability distribution of the data is known (or can be estimated) (Ruszczyński and Shapiro 2003). The approaches focused on the quality of the solution for some data domains (bounded regions) are considered robust (Ben-Tal et al. 2009; Bertsimas and Thiele 2006). Notion of robust solutions was first introduced for statistical decisions in 1964 by Huber (1964). Stochastic programming models with uncertain probability distributions first had been introduced in (Dupacova 1987; Ermoliev et al. 1985). Practical importance of

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the performance sensitivity against data uncertainty and errors has later attracted considerable attention to the search for robust solutions (see (Hampel et al. 1986)). In general decision theory under uncertainty the notion of robustness may have rather broad set of definitions (Ermoliev and Hordijk 2006). The precise concept of robustness depends on the way uncertain data domains and the quality or stability characteristics are introduced.

A conservative notion of robustness focusing on worst case scenario results is widely accepted and the max-min optimization is commonly used to seek robust solutions. Although shortcomings of the worst case approaches are known (Ermoliev and Wets 1988). Recently, a more advanced concept of ordered weighted averaging was introduced into robust optimization (Perny et al. 2006), thus allowing to optimize combined performances under the worst case scenario together with the performances under the second worst scenario, the third worst and so on. Such an approach exploits better the entire distribution of objective vectors in search for robust solutions and, more importantly, it introduces some tools for modeling robust preferences.

In this paper we focus on robust approaches where the probabilities are unknown or imprecise. Having assumed that the probabilities may vary within given intervals, we optimize the worst case expected outcome with respect to the probabilities perturbation set. For the case of unlimited perturbations the worst case expectation becomes the worst outcome (max-min solution). In general case, the worst case expectation is a generalization of the tail mean. Nevertheless, it can be effectively reformulated as a Linear Programming (LP) expansion of the original problem.

The paper is organized as follows. In the next section we recall the tail mean (Conditional Value at Risk, CVaR) solution concept providing a new proof of the LP computational model which remains applicable for more general problems related to the robust solution concepts. Section 3 contains the main results. We show that the robust solution for only upper bounded probabilities is the tail β-mean solution for an appropriate β value. For proportional upper and lower limits on probability perturbation the robust solution may be expressed as optimization of appropriately combined the mean and the tail mean criteria. Finally, a general robust solution for any arbitrary intervals of probabilities or probabilities perturbations can be expressed with optimization problem very similar to the tail β-mean and thereby easily implementable with auxiliary linear inequalities. In Sect. 4 we show how for the specific case of LP problems, alternative dual models of robust solutions may be built to overcome high dimensionality caused by the number of scenarios. The computational advantages of the dual models are demonstrated on the portfolio optimization problem in Sect. 5.

2 Robust Solution Concept

Consider a decision problem under uncertainty where the decision is based on the maximization of a scalar (real valued) outcome. The simplest representation of uncertainty depends on a finite set Ω ($|\Omega| = m$) of predefined scenarios. The final outcome is uncertain and only its realizations under various scenarios $\omega \in \Omega$ are known. Exactly, for each scenario ω the corresponding outcome realization is given as a function of the decision variables $y_\omega = f_\omega(\mathbf{x})$ where $\mathbf{x}$ denotes a vector of decision variables to be selected from the feasible set $Q \subset R^n$ of constraints under consideration. Let us define the set of attainable outcomes $A = \{\mathbf{y} = (y)_{\omega\in\Omega} : y_\omega = f_\omega(\mathbf{x}) \; \forall \; \omega \in \Omega, \quad \mathbf{x} \in Q\}$. We are interested in larger outcomes under each scenario. Hence, the decision under uncertainty can be considered a multiple criteria optimization problem (Haimes 1993; Ogryczak 2002)

$$\max \; \{ \, (y_\omega)_{\omega\in\Omega} \; : \; \mathbf{y} \in A \, \} \, . \tag{1}$$

From the perspective of decision making under uncertainty, the model (1) only specifies that we are interested in maximization of outcomes under all scenarios $\omega \in \Omega$. In order to make the multiple objective model operational for the decision support process, one needs to assume some solution concept well adjusted to the decision maker's preferences.

Within the decision problems under risk it is assumed that the exact values of the underlying scenario probabilities p_ω $(\omega \in \Omega)$ are given or can be estimated. This is a basis for the stochastic programming approaches where the solution concept depends on the maximization of the expected value (the mean outcome)

$$\mu(\mathbf{y}) = \sum_{\omega\in\Omega} y_\omega p_\omega \tag{2}$$

or some risk function. In particular, the risk functions $\mu_{\delta^k}(\mathbf{y}) = \mu(\mathbf{y}) - \delta^k(\mathbf{y})$ based on the downside semideviations

$$\delta^k(\mathbf{y}) = \left[\sum_{\omega\in\Omega} \max\{\mu(\mathbf{y}) - y_\omega p_\omega, 0\}^k \right]^{1/k} \tag{3}$$

are consistent with the second degree stochastic dominance (Ogryczak and Ruszczyński 2001) and thereby coherent (Artzner et al. 1999). Among them, the Mean Absolute Deviation (δ^1) related risk function can be expressed as the mean of downside distribution $\mu_{\delta^1}(\mathbf{y}) = \sum_{\omega\in\Omega} \min\{\mu(\mathbf{y}), y_\omega\} p_\omega$.

Recently, the second order quantile risk measures have been introduced in different ways by many authors (Artzner et al. 1999; Embrechts et al. 1997; Ermoliev and Leonardi 1982; Ogryczak 1999; Rockafellar and Uryasev 2000). They generally represent the (worst) tail mean defined as the mean within the specified tolerance level (quantile) of the worst outcomes. Within the decision under risk literature, and especially related to finance application, the tail mean quantity is usually called Tail VaR, Average VaR or Conditional VaR (where VaR reads after Value-at-Risk) (Pflug 2000). Actually, the name CVaR after (Rockafellar and Uryasev 2000) is now the most commonly used. Although, since we will consider the measure with respect to distributions without a formally defined probabilistic space we will refer to it as the tail mean. The tail mean maximization is consistent with the second degree stochastic dominance (Ogryczak and Ruszczyński 2002) and it meets the requirements of coherent risk measurement (Pflug 2000).

For any probabilities p_ω and tolerance level β the corresponding tail mean can be mathematically formalized as follows (Ogryczak 2002; Ogryczak and Ruszczyński 2002). Having defined the right-continuous cumulative distribution function (cdf): $F_{\mathbf{y}}(\eta) = \text{Prob}[y_w \leq \eta]$, we introduce the quantile function $F_{\mathbf{y}}^{(-1)}$ as the left-continuous inverse of the cumulative distribution function $F_{\mathbf{y}}$:

$$F_{\mathbf{y}}^{(-1)}(\beta) = \inf \{\eta : F_{\mathbf{y}}(\eta) \geq \beta\} \quad \text{for } 0 < \beta \leq 1.$$

By integrating $F_{\mathbf{y}}^{(-1)}$ one gets the (worst) tail mean

$$\mu_\beta(\mathbf{y}) = \frac{1}{\beta} \int_0^\beta F_{\mathbf{y}}^{(-1)}(\alpha) d\alpha \quad \text{for } 0 < \beta \leq 1. \tag{4}$$

the point value of the absolute Lorenz curve (Ogryczak 2000). The latter makes the tail means directly related to the dual theory of choice under risk (Quiggin 1982; Roell 1987; Yaari 1987).

Maximization of the tail β-mean

$$\max_{\mathbf{y} \in A} \mu_\beta(\mathbf{y}) \tag{5}$$

defines the tail β-mean solution concept. When parameter β approaches 0, the tail β-mean tends to the smallest outcome

$$M(\mathbf{y}) = \min\{y_\omega : \omega \in \Omega\} = \lim_{\beta \to 0_+} \mu_\beta(\mathbf{y}).$$

On the other hand, for $\beta = 1$ the corresponding tail mean becomes the standard mean ($\mu_1(\mathbf{y}) = \mu(\mathbf{y})$).

Note that, due to the finite number of scenarios, the tail β-mean is well defined by the following optimization

$$\mu_\beta(\mathbf{y}) = \min_{u_\omega} \left\{ \frac{1}{\beta} \sum_{\omega \in \Omega} y_\omega u_\omega : \sum_{\omega \in \Omega} u_\omega = \beta,\ 0 \leq u_\omega \leq p_\omega \ \forall\, \omega \in \Omega \right\}. \tag{6}$$

Problem (6) is a Linear Program for a given outcome vector $\mathbf{y}$ while it becomes nonlinear for $\mathbf{y}$ being a vector of variables as in the tail β-mean problem (5). It turns out that this difficulty can be overcome by an equivalent LP formulation of the β-mean that allows one to implement the β-mean problem (5) with auxiliary linear inequalities. Namely, the following theorem recalls Rockafellar and Uryasev (2000) LP model for continuous distributions which remains valid for a general distribution (Ogryczak and Ruszczyński 2002). Although we introduce a new proof which can be further generalized for a family of robust solution concepts we consider.

Theorem 1. *For any outcome vector* $\mathbf{y}$ *with the corresponding probabilities* p_ω, *and for any real value* $0 < \beta \leq 1$, *the tail* β*-mean outcome is given by the following linear program:*

$$\mu_\beta(\mathbf{y}) = \max_{t,d_\omega} \left\{ t - \frac{1}{\beta} \sum_{\omega \in \Omega} p_\omega d_\omega : y_\omega \geq t - d_\omega,\ d_\omega \geq 0\ \forall\ \omega \in \Omega \right\}. \tag{7}$$

Proof. The theorem can be proven by taking advantage of the LP dual to (6). Introducing dual variable t corresponding to the equation $\sum_{\omega \in \Omega} u_\omega = \beta$ and variables d_ω corresponding to upper bounds on u_ω one gets the LP dual (7). Due to the duality theory, for any given vector $\mathbf{y}$ the tail β-mean $\mu_\beta(\mathbf{y})$ can be found as the optimal value of the LP problem (7). □

Frequently, scenario probabilities are unknown or imprecise. Uncertainty is then represented by limits (intervals) on possible values of probabilities varying independently (Thiele 2008). We focus on such representation to define robust solution concept. Generally, we consider the case of unknown probabilities belonging to the hypercube:

$$\mathbf{u} \in U = \left\{ (u_1, u_2, \ldots, u_m) : \sum_{\omega \in \Omega} u_\omega = 1,\ \Delta^l_\omega \leq u_\omega \leq \Delta^u_\omega\ \forall\ \omega \in \Omega \right\} \tag{8}$$

where obviously

$$\sum_{\omega \in \Omega} \Delta^l_\omega \leq 1 \leq \sum_{\omega \in \Omega} \Delta^u_\omega.$$

Focusing on the mean outcome as the primary system efficiency measure to be optimized we get the robust mean solution concept

$$\max_{\mathbf{y}} \min_{\mathbf{u}} \left\{ \sum_{\omega \in \Omega} u_\omega y_\omega : \mathbf{u} \in U,\ \mathbf{y} \in A \right\}. \tag{9}$$

Further, taking into account that all the constraints of attainable set A remain unchanged while the probabilities are perturbed, the robust mean solution can be rewritten as

$$\max_{\mathbf{y} \in A} \min_{\mathbf{u} \in U} \sum_{\omega \in \Omega} u_\omega y_\omega = \max_{\mathbf{y} \in A} \left\{ \min_{\mathbf{u} \in U} \sum_{\omega \in \Omega} u_\omega y_\omega \right\} = \max_{\mathbf{y} \in A} \mu^U(\mathbf{y}) \tag{10}$$

where

$$\begin{aligned} \mu^U(\mathbf{y}) &= \min_{\mathbf{u} \in U} \sum_{\omega \in \Omega} u_\omega y_\omega \\ &= \min_{u_\omega} \left\{ \sum_{\omega \in \Omega} y_\omega u_\omega : \sum_{\omega \in \Omega} u_\omega = 1,\ \Delta^l_\omega \leq u_\omega \leq \Delta^u_\omega\ \forall\ \omega \in \Omega \right\} \end{aligned} \tag{11}$$

represent the worst case mean outcomes for given outcome vector $\mathbf{y} \in A$ with respect to the probabilities set U.

Similar robust solution concepts can be built for various risk functions used instead of the mean. For the tail mean (CVaR) optimization, the corresponding robust tail β-mean solution can be expressed as

$$\max_{\mathbf{y}\in A} \mu_\beta^U(\mathbf{y}) \tag{12}$$

where

$$\mu_\beta^U(\mathbf{y}) = \min_{\mathbf{u}\in U} \min_{u'_\omega} \left\{ \frac{1}{\beta} \sum_{\omega\in\Omega} y_\omega u'_\omega : \sum_{\omega\in\Omega} u'_\omega = \beta,\ 0 \le u'_\omega \le u_\omega\ \forall\, \omega \in \Omega \right\}. \tag{13}$$

represents the worst case tail β-mean outcome for given outcome vector $\mathbf{y} \in A$ with respect to the probabilities set U.

3 Tail Mean and Related Robust Solution Concepts

Let us consider first the robust mean solution (10) in the case of unlimited probability perturbations ($\Delta_\omega^l = 0$ and $\Delta_\omega^u = 1$). One may easily notice that the worst case mean outcome (11) becomes the worst outcome

$$\mu^U(\mathbf{y}) = \min_{u_\omega} \left\{ \sum_{\omega\in\Omega} y_\omega u_\omega : \sum_{\omega\in\Omega} u_\omega = 1,\ 0 \le u_\omega \le 1\ \forall\, \omega \in \Omega \right\} = \min_{\omega\in\Omega} y_\omega$$

thus leading to the conservative robust solution concept represented by the max-min approach.

For the case of probabilities lying in a given box with relaxed lower limits ($\Delta_\omega^l = 0\ \forall\ \omega \in \Omega$) the worst case mean outcome (11) becomes the classical tail mean outcome. Hence, the robust solution (10) may be represented as the tail β-mean with respect to appropriately rescaled probabilities.

Theorem 2. *The robust solution the worst case mean outcome (9)–(11) with relaxed lower bounds may be represented as the tail β-mean with respect to probabilities*

$$p_\omega = \Delta_\omega^u \Big/ \sum_{\omega\in\Omega} \Delta_\omega^u \quad \text{and} \quad \beta = 1 \Big/ \sum_{\omega\in\Omega} \Delta_\omega^u,$$

and it can be found by simple expansion of the optimization problem with auxiliary linear constraints and variables to the following:

$$\max_{\mathbf{y},\mathbf{d},t} \left\{ t - \sum_{\omega\in\Omega} \Delta_\omega^u d_\omega : \quad \mathbf{y} \in A; \quad y_\omega \ge t - d_\omega,\ d_\omega \ge 0\ \forall\, \omega \in \Omega \right\}. \tag{14}$$

Proof. Note that by simple rescaling of variables with $s^u = \sum_{\omega\in\Omega} \Delta^u_\omega$ one gets

$$\mu^U(\mathbf{y}) = \min_{u_\omega} \left\{ \sum_{\omega\in\Omega} y_\omega u_\omega : \sum_{\omega\in\Omega} u_\omega = 1,\ 0 \le u_\omega \le \Delta^u_\omega\ \forall\ \omega\in\Omega \right\}$$
$$= \min_{u'_\omega} \left\{ s^u \sum_{\omega\in\Omega} y_\omega u'_\omega : \sum_{\omega\in\Omega} u'_\omega = \frac{1}{s^u}, 0 \le u'_\omega \le \frac{\Delta^u_\omega}{s^u}\ \forall\ \omega\in\Omega \right\}.$$

Hence, the robust solution may be represented as the tail $(1/s^u)$-mean with respect to probabilities $p_\omega = \Delta^u_\omega/s^u$. Following Theorem 1, it can searched by solving (14). □

Note that with $\Delta^u_\omega = 1$ for $\omega\in\Omega$ we represent the robust solution (11) as the tail β-mean with $p_\omega = 1/m$ and $\beta = 1/m$ thus representing the max-min model. In the case of $\Delta^u_\omega = k/m$ for $\omega\in\Omega$ we get $p_\omega = 1/m$ and $\beta = 1/k$. For the specific case of given probabilities $\bar{\mathbf{p}}$ with possible perturbations bounded proportionally it is possible to express the corresponding robust solution (11) as the tail mean based on the original probabilities. Indeed, in the case of $\Delta^u_\omega = (1+\delta^+)\bar{p}_\omega$ we get in Theorem 2

$$p_\omega = \Delta^u_\omega \Big/ \sum_{\omega\in\Omega} \Delta^u_\omega = \bar{p}_\omega.$$

In the general case of possible lower limits, the robust mean solution concept (9)–(11) cannot be directly expressed as an appropriate tail β-mean. It turns out, however, that it can be expressed by the optimization with combined criteria of the tail β-mean and the mean.

Theorem 3. *The robust mean solution concept (9)–(11) is equivalent to the convex combination of the mean and the tail β-mean criteria maximization*

$$\max_{\mathbf{y}\in A} \mu^U(\mathbf{y}) = \max_{\mathbf{y}\in A} \left[\lambda\mu(\mathbf{y}) + (1-\lambda)\mu_\beta(\mathbf{y})\right] \tag{15}$$

with

$$\beta = \left(1 - \sum_{\omega\in\Omega} \Delta^l_\omega\right) \Big/ \sum_{\omega\in\Omega} (\Delta^u_\omega - \Delta^l_\omega) \quad \text{and} \quad \lambda = \sum_{\omega\in\Omega} \Delta^l_\omega,$$

where the tail mean $\mu_\beta(\mathbf{y})$ is defined according to probabilities p'_ω while the mean $\mu(\mathbf{y})$ is considered with respect to probabilities p''_ω:

$$p'_\omega = (\Delta^u_\omega - \Delta^l_\omega) \Big/ \sum_{\omega\in\Omega} (\Delta^u_\omega - \Delta^l_\omega) \quad \text{and} \quad p''_\omega = \Delta^l_\omega \Big/ \sum_{\omega\in\Omega} \Delta^l_\omega \quad \text{for } \omega\in\Omega.$$

Proof. When introducing scaling factors $s^u = \sum_{\omega\in\Omega} \Delta^u_\omega$ and $s^l = \sum_{\omega\in\Omega} \Delta^l_\omega$, the worst case mean outcome (11) can be expressed as follows

$$\begin{aligned}
\mu^U(\mathbf{y}) &= \min_{u_\omega} \left\{ \sum_{\omega\in\Omega} y_\omega u_\omega : \sum_{\omega\in\Omega} u_\omega = 1,\ \Delta^l_\omega \le u_\omega \le \Delta^u_\omega\ \forall\, \omega\in\Omega \right\} \\
&= \min_{u'_\omega} \left\{ \sum_{\omega\in\Omega} y_\omega u'_\omega : \sum_{\omega\in\Omega} u'_\omega = 1 - s^l, 0 \le u'_\omega \le \Delta^u_\omega - \Delta^l_\omega\ \forall\, \omega\in\Omega \right\} \\
&\quad + \sum_{\omega\in\Omega} y_\omega \Delta^u_\omega \\
&= (1-s^l) \min_{u''_\omega} \left\{ \frac{s^u - s^l}{1-s^l} \sum_{\omega\in\Omega} y_\omega u''_\omega : \sum_{\omega\in\Omega} u''_\omega = \frac{1-s^l}{s^u - s^l}, \right. \\
&\qquad \left. 0 \le u''_\omega \le \frac{\Delta^u_\omega - \Delta^l_\omega}{s^u - s^l}\ \forall\, \omega\in\Omega \right\} + s^l \textstyle\sum_{\omega\in\Omega} y_\omega \frac{\Delta^l_\omega}{s^l} \\
&= (1-\lambda)\mu_\beta(\mathbf{y}) + \lambda\mu(\mathbf{y})
\end{aligned}$$

which completes the proof. □

Corollary 1. *The robust mean solution concept (10)–(11) for the specific case of given probabilities* $\bar{\mathbf{p}}$ *with possible perturbations bounded proportionally* $\Delta^l_\omega = (1-\delta^-)\bar{p}_\omega$ *and* $\Delta^u_\omega = (1+\delta^+)\bar{p}_\omega$ *for all* $\omega \in \Omega$ *is equivalent to the convex combination of the mean and tail* β*-mean criteria maximization*

$$\max_{\mathbf{y}\in A} \mu^U(\mathbf{y}) = \max_{\mathbf{y}\in A} \left[\lambda\mu(\mathbf{y}) + (1-\lambda)\mu_\beta(\mathbf{y})\right] \tag{16}$$

with $\beta = \delta^-/(\delta^+ + \delta^-)$ *and* $\lambda = 1 - \delta^-$ *where both the mean* $\mu(\mathbf{y})$ *and the tail mean* $\mu_\beta(\mathbf{y})$ *are calculated with respect to the original probabilities* $\bar{p}_\omega$*.*

Proof. For proportionally bounded perturbations

$$\Delta^l_\omega = (1-\delta^-)\bar{p}_\omega \quad \text{and} \quad \Delta^u_\omega = (1+\delta^+)\bar{p}_\omega$$

formula 15 of Theorem 3 is fulfilled with

$$\beta = \frac{1 - \sum_{\omega\in\Omega} \Delta^l_\omega}{\sum_{\omega\in\Omega} \left(\Delta^u_\omega - \Delta^l_\omega\right)} = \frac{\delta^-}{\delta^+ + \delta^-}$$

and

$$\lambda = \sum_{\omega\in\Omega} \Delta^l_\omega = 1 - \delta^-.$$

Further, where the tail mean is defined according to probabilities

$$p'_\omega = \frac{\Delta^u_\omega - \Delta^l_\omega}{\sum_{\omega\in\Omega}\left(\Delta^u_\omega - \Delta^l_\omega\right)} = \frac{\left(\delta^+ + \delta^-\right)\bar{p}_\omega}{\delta^+ \sum_{\omega\in\Omega} \bar{p}_\omega + \delta^- \sum_{\omega\in\Omega} \bar{p}_\omega} = \bar{p}_\omega$$

as well as the mean is also considered with respect to probabilities

$$p''_\omega = \frac{\Delta^l_\omega}{\sum_{\omega\in\Omega}\Delta^l_\omega} = \frac{\left(1-\delta^l_\omega\right)\bar{p}_\omega}{\left(1-\delta^l_\omega\right)\sum_{\omega\in\Omega}\bar{p}_\omega} = \bar{p}_\omega$$

which completes the proof. □

Alternatively, one can take advantages of the fact that the structure of optimization problem (11) remains very similar to that of the tail β-mean (6). Note that problem (11) is an LP for a given outcome vector $\mathbf{y}$ while it becomes nonlinear for $\mathbf{y}$ being a vector of variables. This difficulty can be overcome similar to Theorem 1 for the tail β-mean.

Theorem 4. *For any arbitrary intervals $[\Delta^l_\omega, \Delta^u_\omega]$ (for all $\omega\in\Omega$) of probabilities, the corresponding robust mean solution (10)–(11) can be given by the following optimization problem*

$$\max_{\mathbf{y},t,d^u_\omega,d^l_\omega}\Big\{t-\sum_{\omega\in\Omega}\Delta^u_\omega d^u_\omega+\sum_{\omega\in\Omega}\Delta^l_\omega d^l_\omega:\quad \mathbf{y}\in A;\quad t-d^u_\omega+d^l_\omega\le y_\omega,\; d^u_\omega,d^l_\omega\ge 0\quad\forall\,\omega\in\Omega\Big\}. \tag{17}$$

Proof. The theorem can be proven by taking advantages of the LP dual to (11). Introducing dual variable t corresponding to the equation $\sum_{\omega\in\Omega}u_\omega = 1$ and variables d^u_ω and d^l_ω corresponding to upper and lower bounds on u_ω, respectively, one gets the following LP dual to problem (11)

$$\mu^U(\mathbf{y}) = \max_{t,d^u_\omega,d^l_\omega}\Big\{t-\sum_{\omega\in\Omega}\Delta^u_\omega d^u_\omega+\sum_{\omega\in\Omega}\Delta^l_\omega d^l_\omega:\quad t-d^u_\omega+d^l_\omega\le y_\omega,\; d^u_\omega,d^l_\omega\ge 0\quad\forall\,\omega\in\Omega\Big\}$$

which completes the proof. □

While considering the tail mean as the basic optimization criterion (CVaR optimization) we have to deal with the robust tail mean solution concepts (12)–(13) to allow for imprecise probabilities. It turns out that this robust solution concept for any arbitrary perturbation set U (8) may be expressed as the standard tail mean with appropriately defined tolerance level and rescaled probabilities.

Theorem 5. *The robust tail β-mean solution (12)–(13) with arbitrary set U (8) may be represented as the tail β'-mean with respect to probabilities*

$$p'_\omega = \Delta^u_\omega\Big/\sum_{\omega\in\Omega}\Delta^u_\omega \quad\textit{and}\quad \beta' = \beta\Big/\sum_{\omega\in\Omega}\Delta^u_\omega,$$

and it can be found by simple expansion of the optimization problem with auxiliary linear constraints and variables to the following:

$$\max_{\mathbf{y},\mathbf{d},t}\left\{t-\frac{1}{\beta}\sum_{\omega\in\Omega}\Delta^u_\omega d_\omega:\quad \mathbf{y}\in A;\quad y_\omega\ge t-d_\omega,\; d_\omega\ge 0\;\forall\,\omega\in\Omega\right\}. \tag{18}$$

Proof. Note that

$$\mu^U_\beta(\mathbf{y}) = \min_{\mathbf{u}\in U}\min_{u'_\omega}\left\{\frac{1}{\beta}\sum_{\omega\in\Omega} y_\omega u'_\omega : \sum_{\omega\in\Omega} u'_\omega = \beta,\ 0 \le u'_\omega \le u_\omega\ \forall\,\omega\in\Omega\right\}$$
$$= \min_{u'_\omega}\left\{\frac{1}{\beta}\sum_{\omega\in\Omega} y_\omega u'_\omega : \sum_{\omega\in\Omega} u'_\omega = \beta,\ 0 \le u'_\omega \le \Delta^u_\omega\ \forall\,\omega\in\Omega\right\}$$

Thus by simple rescaling of variables with $s^u = \sum_{\omega\in\Omega}\Delta^u_\omega$ one gets

$$\mu^U_\beta(\mathbf{y}) = \min_{u''_\omega}\left\{\frac{s^u}{\beta}\sum_{\omega\in\Omega} y_\omega u''_\omega : \sum_{\omega\in\Omega} u''_\omega = \frac{\beta}{s^u}, 0 \le u''_\omega \le \frac{\Delta^u_\omega}{s^u}\ \forall\,\omega\in\Omega\right\}.$$

Hence, the robust solution may be represented as the tail (β/s^u)-mean with respect to probabilities $p_\omega = \Delta^u_\omega/s^u$. Following Theorem 1, it can searched by solving (18). □

Corollary 2. *The robust tail β-mean solution concept (12)–(13) for the specific case of given probabilities $\bar{\mathbf{p}}$ with possible perturbations upper bounded proportionally $\Delta^u_\omega = (1+\delta^+)\bar{p}_\omega$ and arbitrary lower bounded (any $\Delta^l_\omega \le \bar{p}_\omega$) for all $\omega\in\Omega$ is equivalent to the tail β'-mean with respect to probabilities $\bar{\mathbf{p}}$ and $\beta' = \beta/(1+\delta^+)$, and it can be found by simple expansion of the optimization problem with auxiliary linear constraints and variables to the following:*

$$\max_{\mathbf{y},\mathbf{d},t}\left\{t - \frac{1+\delta^+}{\beta}\sum_{\omega\in\Omega}\bar{p}_\omega d_\omega :\quad \mathbf{y}\in A;\quad y_\omega \ge t - d_\omega,\ d_\omega \ge 0\ \forall\,\omega\in\Omega\right\}. \tag{19}$$

4 Dual LP Models

Following (10), the robust mean solution concept is given as

$$\max_{\mathbf{y}\in A}\mu^U(\mathbf{y}) = \max_{\mathbf{y}\in A}\left\{\min_{\mathbf{u}\in U}\sum_{\omega\in\Omega} y_\omega u_\omega\right\} = \max_{\mathbf{y}\in A}\min_{\mathbf{u}\in U}\sum_{\omega\in\Omega} u_\omega y_\omega$$

where the inner optimization problem (11) represents the worst case mean outcome for given outcome vector $\mathbf{y}\in A$ with respect to the probabilities set U. It is an LP for a given vector $\mathbf{y}$ but it turns into nonlinear within the entire robust optimization problem (5), due to the quadratic objective function $\sum_{\omega\in\Omega} y_\omega u_\omega$. This difficulty is overcome by an equivalent dual LP formulation of problem (6). Indeed, introducing dual variable t corresponding to the equation $\sum_{\omega\in\Omega} u_\omega = 1$ and variables d^u_ω and d^l_ω corresponding to upper and lower bounds on u_ω, respectively, we get the following LP dual to problem (11)

$$\mu^U(\mathbf{y}) = \max_{t,d^u_\omega,d^l_\omega} \Big\{ t - \sum_{\omega\in\Omega} \Delta^u_\omega d^u_\omega + \sum_{\omega\in\Omega} \Delta^l_\omega d^l_\omega : \\ t - d^u_\omega + d^l_\omega \le y_\omega,\ d^u_\omega, d^l_\omega \ge 0 \quad \forall\, \omega \in \Omega \Big\} \tag{20}$$

This leads us to the standard LP model (17) of Theorem 4 for the robust optimization. The model dimensionality is strongly affected by the number of scenarios under consideration. The latter may be huge in the case of more advanced simulation models employed for scenario generation (Pflug 2001).

An alternative robust optimization models can be built for LP problems by taking advantages of the minimax theorem. Note that both sets A and U are convex polyhedra. Hence, formula (5) can be rewritten into a dual form

$$\max_{\mathbf{y}\in A} \min_{\mathbf{u}\in U} \sum_{\omega\in\Omega} u_\omega y_\omega = \min_{\mathbf{u}\in U} \max_{\mathbf{y}\in A} \sum_{\omega\in\Omega} u_\omega y_\omega = \min_{\mathbf{u}\in U} D(\mathbf{u}) \tag{21}$$

with the inner optimization problem

$$D(\mathbf{u}) = \max_{\mathbf{y}} \Big\{ \sum_{\omega\in\Omega} u_\omega y_\omega : \mathbf{y} \in A \Big\}. \tag{22}$$

The inner optimization problem although being an LP for a given vector $\mathbf{u}$ has the quadratic objective function $\sum_{\omega\in\Omega} u_\omega y_\omega$ within the entire robust optimization problem (21) where $\mathbf{u}$ is also a vector of variables. Again, this difficulty can be resolved by taking advantages of the LP dual $D^*(\mathbf{u})$ to the inner problem $D^*(\mathbf{u})$. Indeed:

$$\min_{\mathbf{u}\in U} D(\mathbf{u}) = \min_{\mathbf{u}\in U} D^*(\mathbf{u}) \tag{23}$$

but solving the latter problem allows us to use the LP methodology. Moreover, set U has only one equation (structural constraint) which makes the problem $\min_{\mathbf{u}\in U} D^*(\mathbf{u})$ much simpler than those of (20). In the next section we illustrate potential advantages of the alternative (dual) model with the portfolio optimization problem.

5 Portfolio Optimization

The portfolio optimization problem we consider follows the original Markowitz' formulation and is based on a single period model of investment. At the beginning of a period, an investor allocates the capital among various securities, thus assigning a nonnegative weight (share of the capital) to each security. Let $J = \{1, 2, \ldots, n\}$ denote a set of securities considered for an investment. For each security $j \in J$, its rate of return is represented by a random variable R_j with a given mean $\mu_j = \mathbb{E}\{R_j\}$. Further, let $\mathbf{x} = (x_j)_{j=1,\ldots,n}$ denote a vector of decision variables x_j expressing the weights defining a portfolio. The weights must satisfy a set of constraints to represent a portfolio. The simplest way of defining a feasible set Q is by a requirement that the weights must sum to one and they are nonnegative (short sales are not allowed), i.e.

$$Q = \left\{ \mathbf{x} : \sum_{j \in J} x_j = 1, \quad x_j \geq 0 \quad \forall\, j \in J \right\}. \tag{24}$$

Hereafter, we perform detailed analysis for the set Q given with constraints (24). Nevertheless, the presented results can easily be adapted to a general LP feasible set given as a system of linear equations and inequalities, thus allowing one to include short sales, upper bounds on single shares or portfolio structure restrictions which may be faced by a real-life investor.

Each portfolio $\mathbf{x}$ defines a corresponding random variable $R_{\mathbf{x}} = \sum_{j \in J} R_j x_j$ that represents the portfolio rate of return while the expected value can be computed as $\mu(\mathbf{x}) = \sum_{j \in J} \mu_j x_j$. We consider m scenarios $\omega \in \Omega$ with probabilities p_ω. We assume that for each random variable R_j its realization r_j^ω under the scenario ω is known. Typically, the realizations are derived from historical data treating m historical periods as equally probable scenarios ($p_\omega = 1/m$). Although the models we analyze do not take advantages of this simplification. The realizations of the portfolio return $R_{\mathbf{x}}$ are given as

$$y_\omega = \sum_{j \in J} r_j^\omega x_j. \tag{25}$$

Following Theorem 4 and taking into account (25), for any arbitrary intervals $[\Delta_\omega^l, \Delta_\omega^u]$ (for all $\omega \in \Omega$) of probabilities, the corresponding robust portfolio optimization problem (10) can be given by the following LP problem:

$$\begin{aligned} \max_{\mathbf{x},\mathbf{y},t,d_\omega^u,d_\omega^l} \; & t - \sum_{\omega \in \Omega} \Delta_\omega^u d_\omega^u + \sum_{\omega \in \Omega} \Delta_\omega^l d_\omega^l : \\ \text{s.t.} \; & \sum_{j \in J} x_j = 1, \quad x_j \geq 0 \qquad \text{for } j \in J \\ & d_\omega^u - d_\omega^l - t + \sum_{j \in J}^{n} r_j^\omega x_j \geq 0, \; d_\omega^u, d_\omega^l \geq 0 \text{ for } \omega \in \Omega \end{aligned} \tag{26}$$

where t is an unbounded variable.

As a particular case of relaxed lower bounds on scenario probabilities ($\Delta_\omega^l = 0$ $\forall \omega \in \Omega$), following Corollary 2 one gets the classical CVaR portfolio optimization model (Mansini et al. 2003):

$$\begin{aligned} \max_{\mathbf{x},\mathbf{y},t,d_\omega} \; & t - \frac{1}{\beta} \sum_{\omega \in \Omega} p_\omega d_\omega \\ \text{s.t.} \; & \sum_{j \in J} x_j = 1, \quad x_j \geq 0 \qquad \text{for } j \in J \\ & d_\omega - t + \sum_{j \in J} r_j^\omega x_j \geq 0, \; d_\omega \geq 0 \text{ for } \omega \in \Omega \end{aligned} \tag{27}$$

with probabilities $p_\omega = \Delta_\omega^u / \sum_{\omega \in \Omega} \Delta_\omega^u$ and the tolerance level $\beta = 1/\sum_{\omega \in \Omega} \Delta_\omega^u$.

Except from the corresponding portfolio constraints (24), model (27) contains m nonnegative variables d_ω plus single variable t and m corresponding linear inequalities. Hence, its dimensionality is proportional to the number of scenarios m. Exactly, the LP model contains $m + n + 1$ variables and $m + 1$ constraints. It does not cause any computational difficulties for a few hundreds scenarios as in several computational analysis based on historical data (Mansini et al. 2007), However, in the case of more advanced simulation models employed for scenario generation one may get several thousands scenarios (Pflug 2001). This may lead to the LP model (27) with huge number of variables and constraints thus decreasing the computational efficiency of the model.

The dual model (23) allows us to formulate the corresponding robust portfolio optimization problem (10), for any arbitrary intervals of probabilities (8), as the following LP problem:

$$\begin{aligned} &\min_{\mathbf{u},q} q \\ &\text{s.t.}\ \ q - \sum_{\omega\in\Omega} r_j^\omega u_\omega \ge 0 \text{ for } j \in J \\ &\qquad \sum_{\omega\in\Omega} u_\omega = 1 \\ &\qquad \Delta_\omega^l \le u_\omega \le \Delta_\omega^u \quad \text{for } \omega \in \Omega. \end{aligned} \tag{28}$$

For the specific case of the CVaR model (27) representing the case of relaxed lower bounds, the dual model takes the following form:

$$\begin{aligned} &\min_{\mathbf{u},q} q \\ &\text{s.t.}\ \ q - \sum_{\omega\in\Omega} r_j^\omega u_\omega \ge 0 \quad \text{for } j = 1,\dots,n \\ &\qquad \sum_{\omega\in\Omega} u_\omega = 1 \\ &\qquad 0 \le u_\omega \le \frac{p_\omega}{\beta} \quad \text{for } \omega \in \Omega. \end{aligned} \tag{29}$$

The dual LP model contains m variables u_ω, but only $n+1$ constraints (n inequalities and one equation) excluding the simple bounds on u_ω not affecting the problem complexity. Actually, the number of constraints in (29) is proportional to the portfolio size n, thus it is independent from the number of scenarios. Exactly, there are $m + 1$ variables and $n + 1$ constraints. This guarantees a high computational efficiency of the dual model even for very large number of scenarios. Note that possible additional portfolio structure requirements are usually modeled with rather small number of linear constraints thus generating small number of additional variables in the dual model. Certainly, the optimal portfolio shares x_j are not directly represented within the solution vector of problem (29) but they are easily available as the dual variables (shadow prices) for inequalities $q - \sum_{\omega\in\Omega} r_j^\omega u_\omega \ge 0$. Moreover, the dual model (29) may be considered a special case within the general theory of dual representations of coherent measures of risk, following from conjugate duality (Sect. 5 in (Miller and Ruszczyński 2008)).

We have run computational tests (Ogryczak and Śliwiński 2010) on the large scale CVaR portfolio optimization instances developed by Lim et al. (2010). The instances were originally generated from a multivariate normal distribution for 50, 100 or 200 securities with the number of scenarios 50,000. All computations were performed on a PC with the Intel Core i7 2.66GHz processor and 6GB RAM employing the simplex code of the CPLEX 12.1 package. An attempt to solve the primal model (27) with $\beta = 0.05$ resulted in 580, 1443 and 5006 seconds of computation on average, for problems with 50, 100 and 200 securities, respectively. Solving the dual models (29) directly by the primal method (standard CPLEX settings) results in computation times 5.3, 13.6 and 38.9 CPU seconds, respectively. Moreover, the computation times remain very low for various confidence levels (Ogryczak and Śliwiński 2010).

6 Conclusions

We have analyzed the robust mean solution concept where uncertainty is represented by limits (intervals) on possible values of scenario probabilities varying independently. Such an approach, in general, leads to complex optimization models with variable coefficients (probabilities). We have shown, however, that the robust mean solution concepts can be expressed with auxiliary linear inequalities, similar to the tail β-mean solution concept based on maximization of the mean in β portion of the worst outcomes. Actually, the robust mean solution for upper limits on probabilities turns out to be the tail β-mean for an appropriate β value. For upper and lower limits the robust mean solution may be sought by optimization of appropriately combined the mean and the tail mean criteria. Thus, a general robust mean solution for any arbitrary intervals of probabilities can be expressed with optimization problem very similar to the tail β-mean and thereby easily implementable with auxiliary linear inequalities. While considering the tail mean as the basic optimization criterion (CVaR optimization) the corresponding robust solution concept for any arbitrary perturbation set may be expressed as the standard tail mean with appropriately defined tolerance level and rescaled probabilities.

Our analysis has shown that the robust mean solution concept is closely related with the tail mean which is the basic equitable solution concept (Kostreva et al. 2004). It corresponds to recent approaches to the robust optimization based on the equitable optimization (Miettinen et al. 2008; Perny et al. 2006; Takeda and Kanamori 2009). Further study on equitable solution concepts and their relations to robust solutions seems to be a promising research direction. In particular, more complex robust preferences can be modeled by combining with various weights the tail means for larger and smaller perturbations thus leading to the combinations of multiple CVaR measures (Mansini et al. 2007).

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AQ2. In Reference (Ogryczak and Śliwiński 2010) please check the inserted year (2010) is ok.