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ON ROBUST SOLUTIONS TO MULTI-OBJECTIVE LINEAR PROGRAMS

Abstract

In multiple criteria linear programming (MOLP) any efficient solution can be found by the weighting approach with some positive weights allocated to several criteria. The weights settings represent preferences model thus involving impreciseness and uncertainties. The resulting weighted average performance may be lower than expected. Several approaches have been developed to deal with uncertain or imprecise data. In this paper we focus on robust approaches to the weighted averages of criteria where the weights are varying. Assume that the weights may be affected by perturbations varying within given intervals. Note that the weights are normalized and although varying independently they must total to 1. We are interested in the optimization of the worst case weighted average outcome with respect to the weights perturbation set. For the case of unlimited perturbations the worst case weighted average becomes the worst outcome (max-min solution). For the special case of proportional perturbation limits this becomes the conditional average. In general case, the worst case weighted average is a generalization of the conditional average. Nevertheless, it can be effectively reformulated as an LP expansion of the original problem.

Keywords

Multiple criteria, linear programming, robustness, conditional average.

Introduction

In multi-objective linear programming (MOLP) any efficient solution can be found by the weighting approach with some positive weighting of criteria. The weights settings represent preferences and inevitably involve impreciseness and uncertainties causing that the resulting weighted average performance may be lower than expected.

Several approaches have been developed to deal with uncertain or imprecise data in optimization problem. The approaches focused on the quality or on the variation (stability) of the solution for some data domains are considered robust. The notion of robustness applied to decision problems was first introduced by Gupta and Rosenhead [2]. Practical importance of the performance sensitivity against data uncertainty and errors has later attracted considerable attention to the search for robust solutions. Actually, as suggested by Roy [18], the concept of robustness should be applied not only to solutions but, more generally to various assertions and recommendations generated within a decision support process. The precise concept of robustness depends on the way the uncertain data domains and the quality or stability characteristics are introduced. Typically, in robust analysis one does not attribute any probability distribution to represent uncertainties. Data uncertainty is rather represented by non-attributed scenarios. Since one wishes to optimize results under each scenario, robust optimization might be in some sense viewed as a multiobjective optimization problem where objectives correspond to the scenarios. However, despite of many similarities of such robust optimization concepts to multiobjective models, there are also some significant differences [3]. Actually, robust optimization is a problem of optimal distribution of objective values under several scenarios [9] rather than a standard multiobjective optimization model.

A conservative notion of robustness focusing on worst case scenario results is widely accepted and the min-max optimization is commonly used to seek robust solutions. The worst case scenario analysis can be applied either to the absolute values of objectives (the absolute robustness) or to the regret values (the deviational robustness) [6]. The latter, when considered from the multiobjective perspective, represents a simplified reference point approach with the utopian (ideal) objective values for all the scenario used as aspiration levels. Recently, a more advanced concept of ordered weighted averaging was introduced into robust optimization [16], thus, allowing to optimize combined performances under the worst case scenario together with the performances under the second worst scenario, the third worst and so on. Such an approach exploits better the entire distribution of objective vectors in search for robust solutions and, more importantly, it introduces some tools for modeling robust preferences. Actually, while more sophisticated concepts of robust optimization are considered within the area of discrete programming models, only the absolute robustness is usually applied to the majority of decision and design problems.

In this paper we focus on robust approaches to the weighted averages of criteria where the weights are imprecise. Assume that the weights may be affected by perturbations varying within given intervals. Note that the weights are normalized and although varying independently they must total to 1. We are interested in the optimization of the worst case weighted average outcome with respect to the weights perturbation set. For the case of unlimited perturbations the worst case weighted average becomes the worst outcome (max-min solution). For the special case of proportional perturbation limits this becomes the tail average. In general case, the worst case weighted average is a generalization of the tail average. Nevertheless, it can be effectively reformulated as an LP expansion of the original problem.

The paper is organized as follows. In the next section we recall the tail mean (conditional min-max) solution concept providing a new proof of the computational model which remains applicable for more general problems related to the robust solution concepts. Section 2 contains the main results. We show that the robust solution for proportional upper limits on weights perturbations is the tail β-mean solution for an appropriate β value. For proportional upper and lower limits on weights perturbation the robust solution may be expressed as optimization of appropriately combined the mean and the tail mean criteria. Finally, a general robust solution for any arbitrary intervals of weights perturbations can be expressed with optimization problem very similar to the tail β-mean and thereby easily implementable with auxiliary linear inequalities.

1. Solution concepts

Consider a decision problem defined as an optimization problem with m linear objective functions $f_i(\mathbf{x}) = \mathbf{c}^i\mathbf{x}$. They can be either maximized or minimized. When all the objective functions are minimized the problem can be written as follows:

$$\min \{ (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_m(\mathbf{x})) \ : \ \mathbf{x} \in Q \} \tag{1}$$

where $\mathbf{x}$ denotes a vector of decision variables to be selected within the feasible set $Q \subset R^q$, of constraints under consideration and $\mathbf{f}(x) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_m(\mathbf{x}))$ is a vector function that maps the feasible set Q into the criterion space R^m. Let us define the set of attainable outcomes

$$A = \{\mathbf{y} \ : \quad y_i = f_i(\mathbf{x}) \; \forall \, i, \quad \mathbf{x} \in Q\} \tag{2}$$

Model (1) only specifies that we are interested in minimization of all objective functions f_i for $i = 1, 2, \ldots, m$. In order to make the multiple objective model operational for the decision support process, one needs to assume some solution concept well adjusted to the decision maker's preferences. The solution concepts are defined by aggregation functions $a : R^m \rightarrow R$. Thus the multiple criteria problem (1) is replaced with the (scalar) minimization problem

$$\min\ \{a(\mathbf{f}(x))\ :\quad \mathbf{x} \in Q\} \tag{3}$$

The most commonly used aggregation is based on the weighted mean where positive importance weights w_i ($i = 1, 2, \ldots, m$) are allocated to several objectives

$$a(\mathbf{y}) = \sum_{i=1}^{m} y_i w_i \tag{4}$$

The weights are typically normalized to the total 1

$$\bar{w}_i = w_i / \sum_{i=1}^{m} w_i \quad \text{for} \quad i = 1, 2, \ldots, m \tag{5}$$

Note that, in the case of equal weights (all $w_i = 1$), all the normalized weights are given as $\bar{w}_i = 1/m$. Due to positive weights, every optimal solution to the weighted mean aggregation (i.e. problem (3) with the aggregation function (4)) is an efficient solution of the original multiple criteria problem. Moreover, in the case of MOLP problems for any efficient solution $\mathbf{x} \in Q$ there exists a weight vector such that $\mathbf{x}$ is an optimal solution to the corresponding weighted problem [19].

Exactly, for the weighted sum solution concept is defined by minimization of the objective function expressing the mean (average) outcome

$$\mu(\mathbf{y}) = \sum_{i=1}^{m} \bar{w}_i y_i$$

but it is also equivalent to minimization of the total outcome $\sum_{i=1}^{m} w_i y_i$. The min-max solution concept is defined by minimization of the objective function representing the *maximum* (worst) outcome

$$M(\mathbf{y}) = \max_{i=1,\ldots,m} y_i$$

and it is not affected by the objective weights at all.

A natural generalization of the maximum (worst) outcome $M(\mathbf{y})$ is the (worst) tail mean defined as the mean within the specified tolerance level (amount) of the worst outcomes. For the simplest case of equal weights, one may simply define the tail mean $\mu_{\frac{k}{m}}(\mathbf{y})$ as the mean outcome for the k worst-off objectives (or rather k/m portion of the worst objectives). This can be mathematically formalized as follows. First, we introduce the ordering map $\Theta : R^m \rightarrow R^m$ such that

$\Theta(\mathbf{y}) = (\theta_1(\mathbf{y}), \theta_2(\mathbf{y}), \ldots, \theta_m(\mathbf{y}))$, where $\theta_1(\mathbf{y}) \geq \theta_2(\mathbf{y}) \geq \cdots \geq \theta_m(\mathbf{y})$ and there exists a permutation τ of set I such that $\theta_i(\mathbf{y}) = y_{\tau(i)}$ for $i = 1, 2, \ldots, m$. The use of ordered outcome vectors $\Theta(\mathbf{y})$ allows us to focus on distributions of outcomes impartially. Next, the linear cumulative map is applied to ordered outcome vectors to get $\bar{\Theta}(\mathbf{y}) = (\bar{\theta}_1(\mathbf{y}), \bar{\theta}_2(\mathbf{y}), \ldots, \bar{\theta}_m(\mathbf{y}))$ defined as

$$\bar{\theta}_k(\mathbf{y}) = \sum_{i=1}^{k} \theta_i(\mathbf{y}), \quad \text{for} \quad k = 1, 2, \ldots, m. \tag{6}$$

The coefficients of vector $\bar{\Theta}(\mathbf{y})$ express, respectively: the largest outcome, the total of the two largest outcomes, the total of the three largest outcomes, etc. Hence, the *tail* $\frac{k}{m}$*-mean* $\mu_{\frac{k}{m}}(\mathbf{y})$ is given as

$$\mu_{\frac{k}{m}}(\mathbf{y}) = \frac{1}{k}\bar{\theta}_k(\mathbf{y}), \quad \text{for} \quad k = 1, 2, \ldots, m. \tag{7}$$

According to this definition the concept of tail mean is based on averaging restricted to the portion of the worst outcomes. For $\beta = k/m$, the tail β-mean represents the average of the k largest outcomes.

For any set of weights and and tolerance level β the corresponding tail mean can be mathematically formalized as follows [9,11]. First, we introduce the left-continuous right tail cumulative distribution function (cdf):

$$F_{\mathbf{y}}(d) = \sum_{i=1}^{m} \bar{w}_i \kappa_i(d) \quad \text{where} \quad \kappa_i(d) = \begin{cases} 1 & \text{if } y_i \geq d \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

which for any real (outcome) value d provides the measure of outcomes greater or equal to d. Next, we introduce the quantile function $F_{\mathbf{y}}^{(-1)}$ as the right-continuous inverse of the cumulative distribution function $F_{\mathbf{y}}$:

$$F_{\mathbf{y}}^{(-1)}(\beta) = \sup \{\eta : F_{\mathbf{y}}(\eta) \geq \beta\} \quad \text{for} \quad 0 < \beta \leq 1.$$

By integrating $F_{\mathbf{y}}^{(-1)}$ one gets the (worst) tail mean:

$$\mu_\beta(\mathbf{y}) = \frac{1}{\beta} \int_0^\beta F_{\mathbf{y}}^{(-1)}(\alpha) d\alpha \quad \text{for } 0 < \beta \leq 1. \tag{9}$$

Minimization of the tail β-mean

$$\min_{\mathbf{y} \in A} \mu_\beta(\mathbf{y}) \tag{10}$$

defines the tail β-mean solution concept. When parameter β approaches 0, the tail β-mean tends to the largest outcome ($M(\mathbf{y}) = \lim_{\beta\to 0_+} \mu_\beta(\mathbf{y})$). On the other hand, for $\beta = 1$ the corresponding tail mean becomes the standard mean ($\mu_1(\mathbf{y}) = \mu(\mathbf{y})$).

Note that, due to the finite distribution of outcomes y_i ($i = 1, 2, \ldots, m$) in our MOLP problems, the tail β-mean is well defined by the following optimization

$$\mu_\beta(\mathbf{y}) = \frac{1}{\beta} \max_{u_i} \{\sum_{i=1}^m y_i u_i \ : \ \sum_{i=1}^m u_i = \beta,\ 0 \le u_i \le \bar{w}_i \ \forall\, i\}. \tag{11}$$

The above problem is a Linear Program (LP) for a given outcome vector $\mathbf{y}$ while it becomes nonlinear for $\mathbf{y}$ being a vector of variables as in the β-mean problem (10). It turns out that this difficulty can be overcome by an equivalent LP formulation of the β-mean that allows one to implement the β-mean problem (10) with auxiliary linear inequalities. Namely, the following theorem is valid [15]. Although we introduce a new proof which can be further generalized for a family of robust solution concepts we consider.

Theorem 1 *For any outcome vector* $\mathbf{y}$ *with the corresponding objective weights* w_i*, and for any real value* $0 < \beta \le 1$*, the tail* β*-mean outcome is given by the following linear program:*

$$\mu_\beta(\mathbf{y}) = \min_{t, d_i} \{t + \frac{1}{\beta} \sum_{i=1}^m \bar{w}_i d_i \ : \ y_i \le t + d_i,\ d_i \ge 0 \ \forall\, i\}. \tag{12}$$

Proof. The theorem can be proven by taking advantage of the LP dual to problem (11). Introducing dual variable t corresponding to the equation $\sum_{i=1}^m u_i = \beta$ and dual variables d_i corresponding to upper bounds on u_i one gets the LP dual (12). Due to the duality theory, for any given vector $\mathbf{y}$ the tail β-mean $\mu_\beta(\mathbf{y})$ can be found as the optimal value of the LP problem (12). □

Following Theorem 1, the tail β-mean solution can be found as an optimal solution to the optimization problem:

$$\min_{\mathbf{y}, \mathbf{d}, t} \{t + \frac{1}{\beta} \sum_{i=1}^m \bar{w}_i d_i \ : \ \mathbf{y} \in A; \quad y_i \le t + d_i,\ d_i \ge 0 \ \forall\, i\}, \tag{13}$$

or in a more compact form:

$$\min_{\mathbf{y}, t} \{t + \frac{1}{\beta} \sum_{i=1}^m \bar{w}_i (y_i - t)^+ \ : \ \mathbf{y} \in A \},$$

where $(.)^+$ denotes the nonnegative part of a number.

For the special case of equal weights ($w_i = 1/m$ for all $i \in I$) and $\beta = k/m$ one gets the tail k/m-mean. Model (13) takes then the form:

$$\min_{\mathbf{y},t} \{t + \frac{1}{k} \sum_{i=1}^{m} (y_i - t)_+ \ : \quad \mathbf{y} \in A\} \tag{14}$$

where $(.)_+$ denotes the nonnegative part of a number and r_k is an auxiliary (unbounded) variable. The latter, with the necessary adaptation to the location problems, is equivalent to the computational formulation of the k–centrum model introduced in which is the same as the computational formulation of the k-centrum introduced in [14]. Hence, Theorem 1 and model (13) providing an alternative proof of that formulation generalize the k-centrum formulation of [14] allowing to consider weights and arbitrary size parameter β but preserving the simple structure and dimension of the optimization problem. Within the decision under risk literature, and especially related to finance application, the β-mean quantity is usually called tail VaR, average VaR or conditional VaR (where VaR reads after Value-at-Risk) [17].

2. Robust solutions

The weighted mean solution concept is usually very attractive solution concept due to maximizing the system efficiency taking into account objective importance. It is defined as

$$\min_{\mathbf{y}\in A} \{\sum_{i=1}^{m} \bar{w}_i y_i\} \tag{15}$$

However, in practical problems the objective weights may vary. Therefore, a robust solution is sought which performs well under uncertain objective weights.

The simplest representation of uncertainty depends on a number of predefined scenarios $s = 1, \ldots, r$. Let $\bar{w}_i^s$ denote the realization of weight i under scenario s. Then one may seek a robust solution by minimizing the mean outcome under the worst scenario

$$\min_{\mathbf{y}\in A} \max_{s=1,\ldots,r} \{\sum_{i=1}^{m} \bar{w}_i^s y_i\} = \min_{\mathbf{y}\in A}\{z : z \geq \sum_{i=1}^{m} \bar{w}_i^s y_i \ \forall\ s\}$$

or by minimizing the maximum regret [1]

$$\min_{\mathbf{y}\in A} \max_{s=1,\ldots,r} \{\sum_{i=1}^{m} \bar{w}_i^s y_i - \bar{b}^s\} = \min_{\mathbf{y}\in A}\{z : z \geq \sum_{i=1}^{m} \bar{w}_i^s y_i - \bar{b}^s \ \forall\ s\}$$

where $\bar{b}^s$ represent the best value under scenario s

$$\bar{b}^s = \min_{\mathbf{y} \in A} \{\sum_{i=1}^{m} \bar{w}_i^s y_i\}.$$

Frequently, uncertainty is represented by limits (intervals) on possible values of weights varying independently rather than by scenarios for all the weights simultaneously. We focus on such representation to define robust solution concept. Assume that the objective weights $\bar{w}_i$ may be affected by perturbations varying within intervals $[-\delta_i, \Delta_i]$. Note that the weights are normalized and although varying independently they must total to 1. Thus the objective weights belong to the hypercube:

$$\mathbf{u} \in W = \{(u_1, u_2, \ldots, u_m) \; : \; \sum_{i=1}^{m} u_i = 1, \; \bar{w}_i - \delta_i \le u_i \le \bar{w}_i + \Delta_i \; \forall \, i\}.$$

Alternatively one may consider completely independent perturbations of unnormalized weights w_i and normalize they later to define set W. Focusing on the mean outcome as the primary system efficiency measure to be optimized we get the robust mean solution concept

$$\min_{\mathbf{y}} \max_{\mathbf{u}} \{\sum_{i=1}^{m} u_i y_i \; : \quad \mathbf{u} \in W, \; \mathbf{y} \in A\}. \tag{16}$$

Further, taking into account the assumption that all the constraints of attainable set A remain unchanged while the importance weights are perturbed, the robust mean solution can be rewritten as

$$\min_{\mathbf{y} \in A} \max_{\mathbf{u} \in W} \sum_{i=1}^{m} u_i y_i = \min_{\mathbf{y} \in A} \{\max_{\mathbf{u} \in W} \sum_{i=1}^{m} u_i y_i\} = \min_{\mathbf{y} \in A} \mu^w(\mathbf{y}) \tag{17}$$

where

$$\begin{aligned} \mu^w(\mathbf{y}) &= \max_{\mathbf{u} \in W} \sum_{i=1}^{m} u_i y_i \\ &= \max_{u_i} \{\sum_{i=1}^{m} y_i u_i \; : \quad \sum_{i=1}^{m} u_i = 1, \; \bar{w}_i - \delta_i \le u_i \le \bar{w}_i + \Delta_i \; \forall \, i\} \end{aligned} \tag{18}$$

represents the worst case mean outcome for given outcome vector $\mathbf{y} \in A$.

Note that in the case of $\delta_i = \Delta_i = 0$ (no perturbations/uncertainty at all) one gets the standard mean outcome $\mu^w(\mathbf{y}) = \sum_{i=1}^{m} y_i \bar{w}_i$ thus the original mean solution concept. On the other hand, for the case of unlimited perturbations ($\delta_i = \bar{w}_i$ and $\Delta_i = 1 - \bar{w}_i$) the worst case mean outcome

$$\mu^w(\mathbf{y}) = \max_{u_i}\{\sum_{i=1}^m y_i u_i \ : \quad \sum_{i=1}^m u_i = 1,\ 0 \le u_i \le 1\ \forall\ i\} = \max_{i=1,\ldots,m} y_i$$

becomes the worst outcome thus leading to the min-max solution concept.

For the special case of proportional perturbation limits $\delta_i = \delta\bar{w}_i$ and $\Delta_i = \Delta\bar{w}_i$ with positive parameters δ and Δ, one gets

$$\mu^w(\mathbf{y}) = \max_{u_i}\{\sum_{i=1}^m y_i u_i : \sum_{i=1}^m u_i = 1,\ \bar{w}_i(1-\delta) \le u_i \le \bar{w}_i(1+\Delta)\ \forall\ i\} \quad (19)$$

In particular, when lower limits are relaxed ($\delta = 1$) this becomes the classical tail mean outcome [12,15] with $\beta = 1/(1+\Delta)$. Thus the tail mean represents the robust mean solution concept for proportionally upper bounded perturbations.

Theorem 2 *The tail β-mean represents a concept of robust mean solution (17) for proportionally upper bounded perturbations $\Delta_i = \Delta\bar{w}_i$ with $\Delta = (1-\beta)/\beta$ and relaxed the lower ones $\delta_i = \bar{w}_i$ for all $i \in I$.*

Proof. For proportionally bounded upper perturbations $\Delta_i = \Delta\bar{w}_i$ and $\delta_i = \bar{w}_i$ the corresponding worst case mean outcome (18) can be expressed as follows

$$\begin{aligned}\mu^w(\mathbf{y}) &= \max_{u_i}\{\sum_{i=1}^m y_i u_i : \sum_{i=1}^m u_i = 1,\ 0 \le u_i \le \bar{w}_i(1+\Delta)\ \forall\ i\} \\ &= (1+\Delta)\max_{u'_i}\{\sum_{i=1}^m y_i u'_i : \sum_{i=1}^m u'_i = \frac{1}{1+\Delta},\ 0 \le u'_i \le \bar{w}_i\ \forall\ i\} \\ &= (1+\Delta)\mu_{\frac{1}{1+\Delta}}(\mathbf{y})\end{aligned}$$

□

As the tail mean is easily defined by auxiliary LP constraints, the same applies to the robust mean solution concept for proportionally bounded upper perturbations and relaxed the lower ones.

Corollary 1 *The robust mean solution concept (17) for proportionally bounded upper perturbations $\Delta_i = \Delta\bar{w}_i$ and relaxed the lower limits $\delta_i = \bar{w}_i$ for all $i \in I$ can be found by simple expansion of the optimization problem with auxiliary linear constraints and variables to the following:*

$$\min_{\mathbf{y},\mathbf{d},t}\{t + (1+\Delta)\sum_{i=1}^m \bar{w}_i d_i : \quad \mathbf{y} \in A; \quad y_i \le t + d_i,\ d_i \ge 0\ \forall\ i\}. \quad (20)$$

Example 1 Consider the following MOLP problem with two objectives:

$$\min \{(x_1, x_2) : 3x_1 + 5x_2 \geq 36,\ x_1 \geq 2,\ x_2 \geq 3\}. \tag{21}$$

The efficient set for this problem is

$$\{(x_1, x_2) : 3x_1 + 5x_2 = 36,\ x_1 \geq 2,\ x_2 \geq 3\},$$

i.e. the entire line segment between vertices $(2, 6)$ and $(7, 3)$, including both vertices.

Let us assume that the DM preferences has been recognized as represented by equal weights $\bar{w}_1 = \bar{w}_2 = 0.5$ although the weights may actually vary around this values thus belonging to the hypercube:

$$W = \{(u_1, u_2) : \quad u_1 + u_2 = 1,\ 0 \leq u_1 \leq 0.5(1 + \Delta),\ 0 \leq u_2 \leq 0.5(1 + \Delta)\}$$

for some $\Delta > 0$. The ideal weights $\bar{w}$ generate the best efficient solution in the vertex $(2, 6)$. However, for weights $(0.35, 0.65)$ one gets rather the vertex $(7, 3)$ as the best solution. Hence, it is quite natural to look for a robust solution which is based on the worst weights within the set W. Following Corollary 1, such a robust solution can be found by solving the expanded LP problem:

$$\begin{aligned} \min\{t + (1 + \Delta)(0.5d_1 + 0.5d_2) :\ & 3x_1 + 5x_2 \geq 36, \\ & x_1 \geq 2,\ x_2 \geq 3, \\ & x_1 \leq t + d_1,\ x_2 \leq t + d_2, \\ & d_1 \geq 0,\ d_2 \geq 0\}. \end{aligned} \tag{22}$$

In our case, due to only to outcomes and equal weights, one can easily notice that for any (x_1, x_2) the best values of auxiliary variables are defined as $t = \min\{x_1, x_2\}$, $d_1 = x_1 - t$ and $d_2 = x_2 - t$. Hence, $d_1 + d_2 = \max\{x_1, x_2\} - \min\{x_1, x_2\} = |x_1 - x_2|$ and the auxiliary variables can be eliminated leading to the ordered weighted objective [13]

$$\begin{aligned} t + (1 + \Delta)(0.5d_1 + 0.5d_2) &= 0.5(1 + \Delta)\max\{x_1, x_2\} \\ &\quad +0.5(1 - \Delta)\min\{x_1, x_2\} \\ &= 0.5(1 + \Delta)\theta_1(\mathbf{x}) + 0.5(1 - \Delta)\theta_2(\mathbf{x}) \end{aligned}$$

or alternatively to its cumulated form

$$\begin{aligned} t + (1 + \Delta)(0.5d_1 + 0.5d_2) &= \Delta\max\{x_1, x_2\} + 0.5(1 - \Delta)(x_1 + x_2) \\ &= \Delta\bar{\theta}_1(\mathbf{x}) + 0.5(1 - \Delta)\bar{\theta}_2(\mathbf{x}). \end{aligned}$$

Hence, our robust optimization problem (22) can be simplified to the following form:

$$\min\{\Delta \max\{x_1, x_2\} + (1-\Delta)0.5(x_1+x_2) : 3x_1+5x_2 \geq 36,\ x_1 \geq 2,\ x_2 \geq 3\}$$

thus representing a convex combination of the original weighted optimization and the minimax optimization models. One may easily verify that for $\Delta = 0.1$ the optimal vertex $(2, 6)$ remains the corresponding robust solution. On the other hand, for $\Delta = 0.5$ the minimax point $(4.5, 4.5)$ becomes the corresponding robust solution.

Certainly, in the case of unequal weights or especially for more than two criteria the robust optimization problem cannot be simply expressed as a combination of the original weighted aggregation with minimax criterion. Nevertheless, the LP formulation (20) can be effectively solved. ☐

In the general case of proportional perturbation limits (19) the robust mean solution concept cannot be directly expressed as an appropriate tail β-mean. It turns out, however, that it can be expressed by the optimization with combined criteria of the tail β-mean and the original mean as shown in the following theorem.

Theorem 3 *The robust mean solution concept (17) for proportionally bounded perturbations $\Delta_i = \Delta\bar{w}_i$ and $\delta_i = \delta\bar{w}_i$ for all $i \in I$ is equivalent to the convex combination of the mean and tail β-mean criteria minimization*

$$\min_{\mathbf{y}\in A} \mu^w(\mathbf{y}) = \min_{\mathbf{y}\in A}(1+\Delta)[\lambda\mu_\beta(\mathbf{y}) + (1-\lambda)\mu(\mathbf{y})] \tag{23}$$

with $\beta = \delta/(\Delta+\delta)$ and $\lambda = (\Delta+\delta)/(1+\Delta)$.

Proof. For proportionally bounded perturbations $\Delta_i = \Delta\bar{w}_i$ and $\delta_i = \delta\bar{w}_i$ the corresponding worst case mean outcome (18) can be expressed as follows

$$\begin{aligned}
\mu^w(\mathbf{y}) &= \max_{u_i}\{\sum_{i=1}^m y_i u_i : \sum_{i=1}^m u_i = 1,\ \bar{w}_i(1-\delta) \leq u_i \leq \bar{w}_i(1+\Delta)\ \forall\, i\} \\
&= (1+\Delta)\max_{u'_i}\{\sum_{i=1}^m y_i u'_i : \sum_{i=1}^m u'_i = \frac{1}{1+\Delta},\ \bar{w}_i\frac{1-\delta}{1+\Delta} \leq u'_i \leq \bar{w}_i\ \forall\, i\} \\
&= (1+\Delta)\max_{u''_i}\{\sum_{i=1}^m y_i u''_i : \sum_{i=1}^m u''_i = \frac{\delta}{1+\Delta},\ 0 \leq u''_i \leq \bar{w}_i\frac{\Delta+\delta}{1+\Delta}\ \forall\, i\} + \\
&\quad + (1-\delta)\sum_{i=1}^m y_i\bar{w}_i
\end{aligned}$$

$$= (\Delta+\delta)\max_{u_i'''}\{\sum_{i=1}^m y_i u_i''' : \sum_{i=1}^m u_i''' = \frac{\delta}{\Delta+\delta},\ 0 \le u_i''' \le \bar{w}_i\ \forall\, i\}+$$
$$+ (1-\delta)\mu(\mathbf{y})$$
$$= (1+\Delta)[\frac{\Delta+\delta}{1+\Delta}\mu_{\frac{\delta}{\Delta+\delta}}(\mathbf{y}) + \frac{1-\delta}{1+\Delta}\mu(\mathbf{y})]$$

which completes the proof. □

Following Theorems 1 and 3, the robust mean solution concept (17) can be expressed as an LP expansion of the original mean problem.

Corollary 2 *The robust mean solution concept (17) for proportionally bounded perturbations $\Delta_i = \Delta\bar{w}_i$ and $\delta_i = \delta\bar{w}_i$ for all $i \in I$ can be found by simple expansion of the mean problem with auxiliary linear constraints and variables to the following problem:*

$$\min_{\mathbf{y},\mathbf{d},t}\{\ \sum_{i=1}^m \bar{w}_i y_i + \frac{\Delta+\delta}{1-\delta}t + \frac{(\Delta+\delta)^2}{\delta(1-\delta)}\sum_{i=1}^m \bar{w}_i d_i : \quad \mathbf{y}\in A;\quad y_i \le t + d_i,\ d_i \ge 0\ \forall\, i\}. \tag{24}$$

In general case of arbitrary intervals of weights perturbations, the worst case mean outcome (18) cannot be expressed as a tail β-mean or its combination. Nevertheless, the structure of optimization problem (18) remains very similar to that of the tail β-mean (11). Note that problem (18) is an LP for a given outcome vector $\mathbf{y}$ while it becomes nonlinear for $\mathbf{y}$ being a vector of variables. This difficulty can be overcome similar to Theorem 1 for the tail β-mean.

Theorem 4 *For any arbitrary intervals $[-\delta_i, \Delta_i]$ (for all $i \in I$) of weights perturbations, the corresponding worst case mean outcome (18) can be given as*

$$\mu^w(\mathbf{y}) = \min_{t,d_i^u,d_i^l}\{\ t + \sum_{i=1}^m (\bar{w}_i+\Delta_i)d_i^u - \sum_{i=1}^m (\bar{w}_i-\delta_i)d_i^l : \quad t + d_i^u - d_i^l \ge y_i,\ d_i^u, d_i^l \ge 0 \quad \forall\, i\}. \tag{25}$$

Proof. The theorem can be proven by taking advantages of the LP dual to (18). Introducing dual variable t corresponding to the equation $\sum_{i=1}^m u_i = 1$ and variables d_i^u and d_i^l corresponding to upper and lower bounds on u_i, respectively, one gets the following LP dual to problem (18). Due the duality theory, for any given vector $\mathbf{y}$ the worst case mean outcome $\mu^w(\mathbf{y})$ can be found as the optimal value of the LP problem (25). □

Following Theorem 4, the robust mean solution concept (17) can be expressed similar to the tail β-mean with auxiliary linear inequalities expanding the original constraints.

Corollary 3 *For any arbitrary intervals $[-\delta_i, \Delta_i]$ (for all $i \in I$) of weights perturbations, the corresponding robust mean solution (17) can be given by the following optimization problem:*

$$\begin{aligned} \min_{\mathbf{y},t,d_i^u,d_i^l} \{ \quad & t + \sum_{i=1}^{m} (\bar{w}_i + \Delta_i) d_i^u - \sum_{i=1}^{m} (\bar{w}_i - \delta_i) d_i^l : \\ & \mathbf{y} \in A; \quad t + d_i^u - d_i^l \geq y_i,\ d_i^u, d_i^l \geq 0 \quad \forall\, i \}. \end{aligned} \tag{26}$$

Actually, there is a possibility to represent general robust mean solution (17) with optimization problem even more similar to the tail β-mean and thereby with lower number of auxiliary variables than in (26).

Theorem 5 *For any arbitrary intervals $[-\delta_i, \Delta_i]$ (for all $i \in I$) of weights perturbations, the corresponding robust mean solution (17) can be given by the following optimization problem*

$$\begin{aligned} \min_{\mathbf{y},t,d_i} \{ \quad & \sum_{i=1}^{m} (\bar{w}_i - \delta_i) y_i + \bar{\delta} t + \sum_{i=1}^{m} (\Delta_i + \delta_i) d_i : \\ & \mathbf{y} \in A; \quad t + d_i \geq y_i,\ d_i \geq 0 \quad \forall\, i \} \end{aligned} \tag{27}$$

where $\bar{\delta} = \sum_{i=1}^{m} \delta_i$.

Proof. Note that the worst case mean (18) may be transformed as follows

$$\begin{aligned} \mu^w(\mathbf{y}) &= \max_{u_i} \{ \sum_{i=1}^{m} y_i u_i : \sum_{i=1}^{m} u_i = 1,\ \bar{w}_i - \delta_i \leq u_i \leq \bar{w}_i + \Delta_i \ \forall\, i \} \\ &= \max_{u_i'} \{ \sum_{i=1}^{m} y_i u_i' : \sum_{i=1}^{m} u_i' = \sum_{i=1}^{m} \delta_i,\ 0 \leq u_i' \leq \Delta_i + \delta_i \ \forall\, i \} + \\ &\quad + \sum_{i=1}^{m} y_i (\bar{w}_i - \delta_i). \end{aligned} \tag{28}$$

Next, replacing the maximization over variables u_i with the corresponding dual we get

$$\mu^w(\mathbf{y}) = \min_{t,d_i} \{ (\sum_{i=1}^{m} \delta_i) t + \sum_{i=1}^{m} (\Delta_i + \delta_i) d_i : t + d_i \geq y_i,\ d_i \geq 0 \ \forall\, i \} + \sum_{i=1}^{m} (\bar{w}_i - \delta_i) y_i$$

Further, minimization over $\mathbf{y} \in A$ leads us to formula (27) which completes the proof. □

For a special case of arbitrary upper bounds Δ_i and completely relaxed lower bound we get the following result.

Corollary 4 *For any arbitrary upper bounds Δ_i and and relaxed the lower ones $\delta_i = \bar{w}_i$ (for all $i \in I$) on weights perturbations, the corresponding robust mean solution (17) can be given by the following optimization problem*

$$\min_{\mathbf{y},t,d_i} \{t + \sum_{i=1}^{m} (\Delta_i + \bar{w}_i)d_i : \mathbf{y} \in A;\ t + d_i \geq y_i,\ d_i \geq 0 \quad \forall\, i\}. \tag{29}$$

Note that optimization problem (29) is very similar to the tail β-mean model (13). Indeed, in the case of proportional upper limits $\Delta_i = \Delta\bar{w}_i$ (for all $i \in I$ problem (29) simplifies to (20) as stated in Corollary 1.

Concluding remarks

For multiple objective linear programming problems with objective weights the mean solution concept is well suited for system efficiency maximization. However, real-life objective weights inevitably involve errors and uncertainties and thereby the resulting performance may be lower than expected. We have analyzed the robust mean solution concept where weights uncertainty is represented by limits (intervals) on possible values of weights varying independently. Such an approach, in general, leads to complex optimization models with variable coefficients (weights).

We have shown that in the case of the weighted multiple objective linear programming problem the robust mean solution concepts can be expressed with auxiliary linear inequalities, similarly to the tail β-mean solution concept [15] based on minimization of the mean in β portion of the worst outcomes. Actually, the robust mean solution for proportional upper limits on weights perturbations turns out to be the tail β-mean for an appropriate β value. For proportional upper and lower limits on weights perturbation the robust mean solution may be sought by optimization of appropriately combined the mean and the tail mean criteria. Finally, a general robust mean solution for any arbitrary intervals of weights perturbations can be expressed with optimization problem very similar to the tail β-mean and thereby easily implementable with auxiliary linear inequalities.

Our analysis has shown that the robust mean solution concept is closely related with the tail mean which is the basic equitable solution concept. It corresponds to recent approaches to the robust optimization based on the equitable optimization ([7], [16], [5]). Further study on equitable solution concepts and their relations to robust solutions seems to be a promising research direction.

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