FAIR AND EFFICIENT RESOURCE ALLOCATION
Bicriteria Models for Equitable Optimization

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Abstract: Resource allocation problems are concerned with the allocation of limited resources among competing activities so as to achieve the best performances. In systems which serve many usersthere is a need to respect some fairness rules while looking for the overall efficiency. The so-called Max-Min Fairness is widely used to meet these goals. However, allocating the resource to optimize the worst performance may cause a dramatic worsening of the overall system efficiency. Therefore, several other fair allocation schemes are searched and analyzed. In this paper we focus on mean-equity approaches which quantify the problem in a lucid form of two criteria: the mean outcome representing the overall efficiency and a scalar measure of inequality of outcomes to represent the equity (fairness) aspects. The mean-equity model is appealing to decision makers and allows a simple trade-off analysis. On the other hand, for typical dispersion indices used as inequality measures, the mean-equity approach may lead to inferior conclusions with respect to the outcomes maximization (system efficiency). Some inequality measures, however, can be combined with the mean itself into optimization criteria that remain in harmony with both inequality minimization and maximization of outcomes. In this paper we introduce general conditions for inequality measures sufficient to provide such an equitable consistency. We verify the conditions for the basic inequality measures thus showing how they can be used not leading to inferior distributions of system outcomes.

1 INTRODUCTION

Resource allocation problems are concerned with the allocation of limited resources among competing activities (Ibaraki and Katoh, 1988). In this paper, we focus on approaches that, while allocating resources to maximize the system efficiency, they also attempt to provide a fair treatment of all the competing activities (Luss, 1999). The problems of efficient and fair resource allocation arise in various systems which serve many users, like in telecommunication systems among others. In networking a central issue is how to allocate bandwidth to flows efficiently and fairly (Bonald and Massoulie, 2001; Denda et al., 2000; Kleinberg et al., 2001; Pióro and Medhi, 2004). In location analysis of public services, the decisions often concern the placement of a service center or another facility in a position so that the users are treated fairly in an equitable way, relative to certain criteria (Ogryczak, 2000). Recently, several research publications relating the fairness and equity concepts to the multiple criteria optimization methodology have appeared (Kostreva et al., 2004; Luss, 1999).

The generic resource allocation problem may be stated as follows. Each activity is measured by an individual performance function that depends on the corresponding resource level assigned to that activity. A larger function value is considered better, like the performance measured in terms of quality level, capacity, service amount available, etc. Models with an (aggregated) objective function that maximizes the mean (or simply the sum) of individual performances are widely used to formulate resource allocation problems, thus defining the so-called mean solution concept. This solution concept is primarily concerned with the overall system efficiency. As based on averaging, it often provides solution where some smaller services are discriminated in terms of allocated resources. An alternative approach depends on the so-called Max-Min solution concept, where the worst performance is maximized. The Max-Min approach is consistent with Rawlsian (Rawls, 1971) theory of justice, especially when additionally regularized with the lexicographic order. The latter is called the Max-

Min Fairness (MMF) and commonly used in networking (Pióro and Medhi, 2004; Ogryczak et al., 2005). Allocating the resources to optimize the worst performances may cause, however, a large worsening of the overall (mean) performances. Therefore, there is a need to seek a compromise between the two extreme approaches discussed above.

Fairness is, essentially, an abstract socio-political concept that implies impartiality, justice and equity (Rawls and Kelly, 2001; Young, 1994), Nevertheless, fairness was frequently quantified with the so-called inequality measures to be minimized (Atkinson, 1970; Rothschild and Stiglitz, 1973). Unfortunately, direct minimization of typical inequality measures contradicts the maximization of individual outcomes and it may lead to inferior decisions. In order to ensure fairness in a system, all system entities have to be equally well provided with the system's services. This leads to concepts of fairness expressed by the equitable efficiency (Kostreva and Ogryczak, 1999; Luss, 1999). The concept of equitably efficient solution is a specific refinement of the Pareto-optimality taking into account the inequality minimization according to the Pigou-Dalton approach. In this paper the use of scalar inequality measures in bicriteria models to search for fair and efficient allocations is analyzed. There is shown that properties of convexity and positive homogeneity together with some boundedness condition are sufficient for a typical inequality measure to guarantee that it can be used consistently with the equitable optimization rules.

2 EQUITY AND FAIRNESS

The generic resource allocation problem may be stated as follows. There is a system dealing with a set I of m services. There is given a measure of services realization within a system. In applications we consider, the measure usually expresses the service quality. In general, outcomes can be measured (modeled) as service time, service costs, service delays as well as in a more subjective way. There is also given a set Q of allocation patterns (allocation decisions). For each service $i \in I$ a function $f_i(\mathbf{x})$ of the allocation pattern $\mathbf{x} \in Q$ has been defined. This function, called the individual objective function, measures the outcome (effect) $y_i = f_i(\mathbf{x})$ of allocation $\mathbf{x}$ pattern for service i. In typical formulations a larger value of the outcome means a better effect (higher service quality or client satisfaction). Otherwise, the outcomes can be replaced with their complements to some large number. Therefore, without loss of generality, we can assume that each individual outcome y_i is to be maximized which allows us to view the generic resource allocation problem as a vector maximization model:

$$\max \{\mathbf{f}(\mathbf{x}) \ : \ \mathbf{x} \in Q\} \tag{1}$$

where $\mathbf{f}(\mathbf{x})$ is a vector-function that maps the decision space $X = R^n$ into the criterion space $Y = R^m$, and $Q \subset X$ denotes the feasible set.

Model (1) only specifies that we are interested in maximization of all objective functions f_i for $i \in I = \{1, 2, \ldots, m\}$. In order to make it operational, one needs to assume some solution concept specifying what it means to maximize multiple objective functions. The solution concepts may be defined by properties of the corresponding preference model. The preference model is completely characterized by the relation of weak preference, denoted hereafter with $\succeq$. Namely, the corresponding relations of strict preference $\succ$ and indifference $\cong$ are defined by the following formulas:

$$\mathbf{y}' \succ \mathbf{y}'' \quad \Leftrightarrow \quad (\mathbf{y}' \succeq \mathbf{y}'' \quad \text{and} \quad \mathbf{y}'' \not\succeq \mathbf{y}'),$$
$$\mathbf{y}' \cong \mathbf{y}'' \quad \Leftrightarrow \quad (\mathbf{y}' \succeq \mathbf{y}'' \quad \text{and} \quad \mathbf{y}'' \succeq \mathbf{y}').$$

The standard preference model related to the Pareto-optimal (efficient) solution concept assumes that the preference relation $\succeq$ is *reflexive*:

$$\mathbf{y} \succeq \mathbf{y}, \tag{2}$$

transitive:

$$(\mathbf{y}' \succeq \mathbf{y}'' \quad \text{and} \quad \mathbf{y}'' \succeq \mathbf{y}''') \ \Rightarrow \ \mathbf{y}' \succeq \mathbf{y}''', \tag{3}$$

and *strictly monotonic*:

$$\mathbf{y} + \varepsilon \mathbf{e}_i \succ \mathbf{y} \quad \text{for } \varepsilon > 0; \ i = 1, \ldots, m, \tag{4}$$

where $\mathbf{e}_i$ denotes the i–th unit vector in the criterion space. The last assumption expresses that for each individual objective function more is better (maximization). The preference relations satisfying axioms (2)–(4) are called hereafter *rational preference relations*. The rational preference relations allow us to formalize the Pareto-optimality (efficiency) concept with the following definitions. We say that outcome vector $\mathbf{y}'$ rationally dominates $\mathbf{y}''$ ($\mathbf{y}' \succ_r \mathbf{y}''$), iff $\mathbf{y}' \succ \mathbf{y}''$ for all rational preference relations $\succeq$. We say that feasible solution $\mathbf{x} \in Q$ is a *Pareto-optimal (efficient)* solution of the multiple criteria problem (1), iff $\mathbf{y} = \mathbf{f}(\mathbf{x})$ is rationally nondominated.

Simple solution concepts for multiple criteria problems are defined by aggregation (or utility) functions $g : Y \to R$ to be maximized. Thus the multiple criteria problem (1) is replaced with the maximization problem

$$\max \{g(\mathbf{f}(\mathbf{x})) \ : \ \mathbf{x} \in Q\} \tag{5}$$

In order to guarantee the consistency of the aggregated problem (5) with the maximization of all individual objective functions in the original multiple criteria problem (or Pareto-optimality of the solution), the aggregation function must be strictly increasing with respect to every coordinate.

The simplest aggregation functions commonly used for the multiple criteria problem (1) are defined as the mean (average) outcome

$$\mu(\mathbf{y}) = \frac{1}{m}\sum_{i=1}^{m} y_i \tag{6}$$

or the worst outcome

$$M(\mathbf{y}) = \min_{i=1,\ldots,m} y_i. \tag{7}$$

The mean (6) is a strictly increasing function while the minimum (7) is only nondecreasing. Therefore, the aggregation (5) using the sum of outcomes always generates a Pareto-optimal solution while the maximization of the worst outcome may need some additional refinement. The mean outcome maximization is primarily concerned with the overall system efficiency. As based on averaging, it often provides a solution where some services are discriminated in terms of performances. On the other hand, the worst outcome maximization, ie, the so-called Max-Min solution concept is regarded as maintaining equity. Indeed, in the case of a simplified resource allocation problem with the knapsack constraints, the Max-Min solution meets the perfect equity requirement. In the general case, with possibly more complex feasible set structure, this property is not fulfilled. Nevertheless, if the perfectly equilibrated outcome vector $\bar{y}_1 = \bar{y}_2 = \ldots = \bar{y}_m$ is nondominated, then it is the unique optimal solution of the corresponding Max-Min optimization problem. In other words, the perfectly equilibrated outcome vector is a unique optimal solution of the Max-Min problem if one cannot find any (possibly not equilibrated) vector with improved at least one individual outcome without worsening any others. Unfortunately, it is not a common case and, in general, the optimal set to the Max-Min aggregation may contain numerous alternative solutions including dominated ones. The Max-Min solution may be then regularized according to the Rawlsian principle of justice (Rawls, 1971) which leads us to the lexicographic Max-Min concepts or the so-called Max-Min Fairness (Marchi and Oviedo, 1992; Ogryczak and Śliwiński, 2006).

In order to ensure fairness in a system, all system entities have to be equally well provided with the system's services. This leads to concepts of fairness expressed by the equitable rational preferences (Kostreva and Ogryczak, 1999). First of all, the fairness requires impartiality of evaluation, thus focusing on the distribution of outcome values while ignoring their ordering. That means, in the multiple criteria problem (1) we are interested in a set of outcome values without taking into account which outcome is taking a specific value. Hence, we assume that the preference model is impartial (anonymous, symmetric). In terms of the preference relation it may be written as the following axiom

$$(y_{\pi(1)},\ldots,y_{\pi(m)}) \cong (y_1,\ldots,y_m) \quad \forall \pi \in \Pi(I) \tag{8}$$

where $\Pi(I)$ denotes the set of all permutations of I. This means that any permuted outcome vector is indifferent in terms of the preference relation. Further, fairness requires equitability of outcomes which causes that the preference model should satisfy the (Pigou–Dalton) principle of transfers. The principle of transfers states that a transfer of any small amount from an outcome to any other relatively worse–off outcome results in a more preferred outcome vector. As a property of the preference relation, the principle of transfers takes the form of the following axiom

$$\mathbf{y} - \varepsilon\mathbf{e}_i + \varepsilon\mathbf{e}_j \succ \mathbf{y} \quad \text{for } 0 < \varepsilon < y_i - y_j \tag{9}$$

The rational preference relations satisfying additionally axioms (8) and (9) are called hereafter *fair (equitable) rational preference relations*. We say that outcome vector $\mathbf{y}'$ *fairly (equitably) dominates* $\mathbf{y}''$ ($\mathbf{y}' \succ_e \mathbf{y}''$), iff $\mathbf{y}' \succ \mathbf{y}''$ for all fair rational preference relations $\succeq$. In other words, $\mathbf{y}'$ fairly dominates $\mathbf{y}''$, if there exists a finite sequence of vectors $\mathbf{y}^j$ ($j = 1,2,\ldots,s$) such that $\mathbf{y}^1 = \mathbf{y}''$, $\mathbf{y}^s = \mathbf{y}'$ and $\mathbf{y}^j$ is constructed from $\mathbf{y}^{j-1}$ by application of either permutation of coordinates, equitable transfer, or increase of a coordinate. An allocation pattern $\mathbf{x} \in Q$ is called *fairly (equitably) efficient* or simply *fair* if $\mathbf{y} = \mathbf{f}(\mathbf{x})$ is fairly nondominated. Note that each fairly efficient solution is also Pareto-optimal, but not vice verse.

In order to guarantee fairness of the solution concept (5), additional requirements on the class of aggregation (utility) functions must be introduced. In particular, the aggregation function must be additionally symmetric (impartial), i.e. for any permutation π of I,

$$g(y_{\pi(1)}, y_{\pi(2)},\ldots,y_{\pi(m)}) = g(y_1,y_2,\ldots,y_m) \tag{10}$$

as well as be equitable (to satisfy the principle of transfers)

$$g(y_1,\ldots,y_i - \varepsilon,\ldots,y_j + \varepsilon,\ldots,y_m) > g(y_1,\ldots,y_m) \tag{11}$$

for any $0 < \varepsilon < y_{i'} - y_{i''}$. In the case of a strictly increasing function satisfying both the requirements (10) and (11), we call the corresponding problem (5) a *fair (equitable) aggregation* of problem (1). Every optimal solution to the fair aggregation (5) of a multiple criteria problem (1) defines some fair (equitable) solution.

Note that both the simplest aggregation functions, the sum (6) and the minimum (7), are symmetric although they do not satisfy the equitability requirement (11). To guarantee the fairness of solutions, some enforcement of concave properties is required. For any strictly concave, increasing utility function $u : R \to R$, the function $g(\mathbf{y}) = \sum_{i=1}^{m} u(y_i)$ is a strictly monotonic and equitable thus defining a family of the fair aggregations. Various concave utility functions u can be used to define such fair solution concepts. In the case of the outcomes restricted to positive values, one may use logarithmic function thus resulting in the *Proportional Fairness* (PF) solution concept (Kelly et al., 1997). Actually, it corresponds to the so-called Nash criterion which maximizes the product of additional utilities compared to the status quo. For a common case of upper bounded outcomes $y_i \le y^*$ one may maximize power functions $-\sum_{i=1}^{m} (y^* - y_i)^p$ for $1 < p < \infty$ which corresponds to the minimization of the corresponding p-norm distances from the common upper bound y^* (Kostreva et al., 2004).

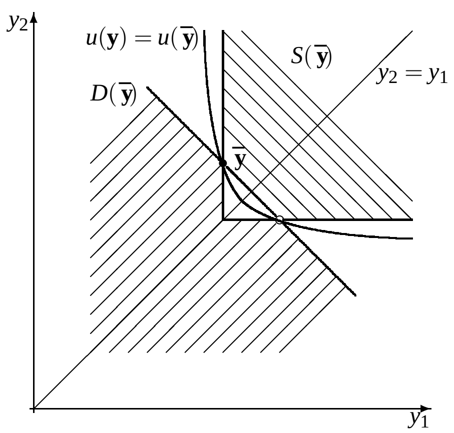

Figure 1: Structure of the fair dominance: $D(\overline{\mathbf{y}})$ – the set fairly dominated by $\overline{\mathbf{y}}$, $S(\overline{\mathbf{y}})$ – the set of outcomes fairly dominating $\overline{\mathbf{y}}$.

Fig. 1 presents the structure of fair dominance for two-dimensional outcome vectors. For any outcome vector $\overline{\mathbf{y}}$, the fair dominance relation distinguishes set $D(\overline{\mathbf{y}})$ of dominated outcomes (obviously worse for all fair rational preferences) and set $S(\overline{\mathbf{y}})$ of dominating outcomes (obviously better for all fair rational preferences). However, some outcome vectors are left (in white areas) and they can be differently classified by various specific fair rational preferences. The MMF fairness assigns the entire interior of the inner white triangle to the set of preferred outcomes while classifying the interior of the external open triangles as worse outcomes. Isolines of various utility functions split the white areas in different ways. One may notice that the set $D(\overline{\mathbf{y}})$ of directions leading to outcome vectors being dominated by a given $\overline{\mathbf{y}}$ is, in general, not a cone and it is not convex. Although, when we consider the set $S(\overline{\mathbf{y}})$ of directions leading to outcome vectors dominating given $\overline{\mathbf{y}}$ we get a convex set.

3 INEQUALITY MEASURES AND FAIR CONSISTENCY

Inequality measures were primarily studied in economics while recently they become very popular tools in Operations Research. Typical inequality measures are some deviation type dispersion characteristics. They are *translation invariant* in the sense that $\rho(\mathbf{y} + \nu\mathbf{e}) = \rho(\mathbf{y})$ for any outcome vector $\mathbf{y}$ and real number ν (where $\mathbf{e}$ vector of units $(1,\ldots,1)$), thus being not affected by any shift of the outcome scale. Moreover, the inequality measures are also *inequality relevant* which means that they are equal to 0 in the case of perfectly equal outcomes while taking positive values for unequal ones.

The simplest inequality measures are based on the absolute measurement of the spread of outcomes, like the *mean absolute difference*

$$\Gamma(\mathbf{y}) = \frac{1}{2m^2} \sum_{i=1}^{m} \sum_{j=1}^{m} |y_i - y_j| \tag{12}$$

or the *maximum absolute difference*

$$d(\mathbf{y}) = \max_{i,j=1,\ldots,m} |y_i - y_j| \tag{13}$$

In most application frameworks better intuitive appeal may have inequality measures related to deviations from the mean outcome like the mean absolute deviation

$$\delta(\mathbf{y}) = \frac{1}{m} \sum_{i=1}^{m} |y_i - \mu(\mathbf{y})| \tag{14}$$

or the *maximum absolute deviation*

$$R(\mathbf{y}) = \max_{i \in I} |y_i - \mu(\mathbf{y})| \tag{15}$$

Note that the *standard deviation* σ (or the *variance* σ^2) represents both the deviations and the spread measurement as

$$\sigma^2(\mathbf{y}) = \frac{\sum_{i \in I} (y_i - \mu(\mathbf{y}))^2}{m} = \frac{\sum_{i \in I} \sum_{j \in I} (y_i - y_j)^2}{2m^2} \tag{16}$$

Deviational measures may be focused on the downside semideviations as related to worsening of outcome while ignoring upper semideviations related to improvement of outcome. One may define the *maximum (downside) semideviation*

$$\Delta(\mathbf{y}) = \max_{i \in I} (\mu(\mathbf{y}) - y_i) \tag{17}$$

and the *mean (downside) semideviation*

$$\bar{\delta}(\mathbf{y}) = \frac{1}{m}\sum_{i\in I}(\mu(\mathbf{y}) - y_i)_+ \tag{18}$$

where $(.)_+$ denotes the nonnegative part of a number. Similarly, the *standard (downside) semideviation* is given as

$$\bar{\sigma}(\mathbf{y}) = \sqrt{\frac{1}{m}\sum_{i\in I}(\mu(\mathbf{y}) - y_i)_+^2} \tag{19}$$

In economics one usually considers relative inequality measures normalized by mean outcome. Among many inequality measures perhaps the most commonly accepted by economists is the Gini coefficient, which is the relative mean difference. One can easily notice that direct minimization of typical inequality measures (especially the relative ones) may contradict the optimization of individual outcomes resulting in equal but very low outcomes. As some resolution one may consider a bicriteria mean-equity model:

$$\max\ \{(\mu(\mathbf{f}(\mathbf{x})), -\rho(\mathbf{f}(\mathbf{x})))\ :\ \mathbf{x}\in Q\} \tag{20}$$

which takes into account both the efficiency with optimization of the mean outcome $\mu(\mathbf{y})$ and the equity with minimization of an inequality measure $\rho(\mathbf{y})$. For typical inequality measures bicriteria model (20) is computationally very attractive since both the criteria are concave and LP implementable for many measures. Unfortunately, for any dispersion type inequality measures the bicriteria mean-equity model is not consistent with the outcomes maximization, and therefore is not consistent with the fair dominance. When considering a simple discrete problem with two allocation patterns P1 and P2 generating outcome vectors $\mathbf{y}' = (0,0)$ and $\mathbf{y}'' = (2,8)$, respectively, for any dispersion type inequality measure one gets $\rho(\mathbf{y}'') > 0 = \rho(\mathbf{y}')$ while $\mu(\mathbf{y}'') = 5 > 0 = \mu(\mathbf{y}')$. Hence, $\mathbf{y}''$ is not bicriteria dominated by $\mathbf{y}'$ and vice versa. Thus for any dispersion type inequality measure ρ, allocation P1 with obviously worse outcome vector than that for allocation P2 is a Pareto-optimal solution in the corresponding bicriteria mean-equity model (20).

Note that the lack of consistency of the mean-equity model (20) with the outcomes maximization applies also to the case of the maximum semideviation $\Delta(\mathbf{y})$ (17) used as an inequality measure whereas subtracting this measure from the mean $\mu(\mathbf{y}) - \Delta(\mathbf{y}) = M(\mathbf{y})$ results in the worst outcome and thereby the first criterion of the MMF model. In other words, although a direct use of the maximum semideviation in the mean-equity model may contradict the outcome maximization, the measure can be used complementary to the mean leading us to the worst outcome criterion which does not contradict the outcome maximization. This construction can be generalized for various (dispersion type) inequality measures. Moreover, we allow the measures to be scaled with any positive factor $\alpha > 0$, in order to avoid creation of new inequality measures as one could consider $\rho_\alpha(X) = \alpha\rho(X)$ as a different inequality measure. For any inequality measure ρ we introduce the corresponding underachievement function defined as the difference of the mean outcome and the (scaled) inequality measure itself, i.e.

$$M_{\alpha\rho}(\mathbf{y}) = \mu(\mathbf{y}) - \alpha\rho(\mathbf{y}). \tag{21}$$

This allows us to replace the original mean-equity bicriteria optimization (20) with the following bicriteria problem:

$$\max\{(\mu(\mathbf{f}(\mathbf{x})), \mu(\mathbf{f}(\mathbf{x})) - \alpha\rho(\mathbf{f}(\mathbf{x})))\ :\ \mathbf{x}\in Q\} \tag{22}$$

where the second objective represents the corresponding underachievement measure $M_{\alpha\rho}(\mathbf{y})$ (21). Note that for any inequality measure $\rho(\mathbf{y}) \geq 0$ one gets $M_{\alpha\rho}(\mathbf{y}) \leq \mu(\mathbf{y})$ thus really expressing underachievements (comparing to mean) from the perspective of outcomes being maximized.

We will say that an inequality measure ρ is *fairly α-consistent* if

$$\mathbf{y}' \succeq_e \mathbf{y}'' \ \Rightarrow\ \mu(\mathbf{y}') - \alpha\rho(\mathbf{y}') \geq \mu(\mathbf{y}'') - \alpha\rho(\mathbf{y}'') \tag{23}$$

The relation of fair α-consistency will be called *strong* if, in addition to (23), the following holds

$$\mathbf{y}' \succ_e \mathbf{y}'' \ \Rightarrow\ \mu(\mathbf{y}') - \alpha\rho(\mathbf{y}') > \mu(\mathbf{y}'') - \alpha\rho(\mathbf{y}''). \tag{24}$$

Theorem 1. *If the inequality measure $\rho(\mathbf{y})$ is fairly α-consistent (23), then except for outcomes with identical values of $\mu(\mathbf{y})$ and $\rho(\mathbf{y})$, every efficient solution of the bicriteria problem (22) is a fairly efficient allocation pattern. In the case of strong consistency (24), every allocation pattern $\mathbf{x} \in Q$ efficient to (22) is, unconditionally, fairly efficient.*

Proof. Let $\mathbf{x}^0 \in Q$ be an efficient solution of (22). Suppose that $\mathbf{x}^0$ is not fairly efficient. This means, there exists $\mathbf{x} \in Q$ such that $\mathbf{y} = \mathbf{f}(\mathbf{x}) \succ_e \mathbf{y}^0 = \mathbf{f}(\mathbf{x}^0)$. Then, it follows $\mu(\mathbf{y}) \geq \mu(\mathbf{y}^0)$, and simultaneously $\mu(\mathbf{y}) - \alpha\rho(\mathbf{y}) \geq \mu(\mathbf{y}^0) - \alpha\rho(\mathbf{y}^0)$, by virtue of the fair α-consistency (23). Since $\mathbf{x}^0$ is efficient to (22) no inequality can be strict, which implies $\mu(\mathbf{y}) = \mu(\mathbf{y}^0)$ and and $\rho(\mathbf{y}) = \rho(\mathbf{y}^0)$.

In the case of the strong fair α-consistency (24), the supposition $\mathbf{y} = \mathbf{f}(\mathbf{x}) \succ_e \mathbf{y}^0 = \mathbf{f}(\mathbf{x}^0)$ implies $\mu(\mathbf{y}) \geq \mu(\mathbf{y}^0)$ and $\mu(\mathbf{y}) - \alpha\rho(\mathbf{y}) > \mu(\mathbf{y}^0) - \alpha\rho(\mathbf{y}^0)$ which contradicts the efficiency of $\mathbf{x}^0$ with respect to (22). Hence, the allocation pattern $\mathbf{x}^0$ is fairly efficient. □

4 FAIR CONSISTENCY CONDITIONS

Typical dispersion type inequality measures are convex, i.e. $\rho(\lambda\mathbf{y}' + (1-\lambda)\mathbf{y}'') \leq \lambda\rho(\mathbf{y}') + (1-\lambda)\rho(\mathbf{y}'')$ for any $\mathbf{y}', \mathbf{y}''$ and $0 \leq \lambda \leq 1$. Certainly, the underachievement function $M_{\alpha\rho}(\mathbf{y})$ must be also monotonic for the fair consistency which enforces more restrictions on the inequality measures. We will show further that convexity together with positive homogeneity and some boundedness of an inequality measure is sufficient to guarantee monotonicity of the corresponding underachievement measure and thereby to guarantee the fair α-consistency of inequality measure itself.

We say that (dispersion type) inequality measure $\rho(\mathbf{y}) \geq 0$ is Δ-*bounded* if it is upper bounded by the maximum downside deviation, i.e.,

$$\rho(\mathbf{y}) \leq \Delta(\mathbf{y}) \quad \forall \mathbf{y}. \tag{25}$$

Moreover, we say that $\rho(\mathbf{y}) \geq 0$ is strictly Δ-bounded if inequality (25) is a strict bound, except from the case of perfectly equal outcomes, i.e., $\rho(\mathbf{y}) < \Delta(\mathbf{y})$ for any $\mathbf{y}$ such that $\Delta(\mathbf{y}) > 0$.

Theorem 2. *Let $\rho(\mathbf{y}) \geq 0$ be a convex, positively homogeneous and translation invariant (dispersion type) inequality measure. If $\alpha\rho(\mathbf{y})$ is Δ-bounded, then $\rho(\mathbf{y})$ is fairly α-consistent in the sense of (23).*

Proof. The relation of fair dominance $\mathbf{y}' \succeq_e \mathbf{y}''$ denotes that there exists a finite sequence of vectors $\mathbf{y}^0 = \mathbf{y}'', \mathbf{y}^1, \ldots, \mathbf{y}^t$ such that $\mathbf{y}^k = \mathbf{y}^{k-1} - \varepsilon_k \mathbf{e}_{i'} + \varepsilon_k \mathbf{e}_{i''}$, $0 \leq \varepsilon_k \leq y_{i'}^{k-1} - y_{i''}^{k-1}$ for $k = 1, 2, \ldots, t$ and there exists a permutation π such that $y'_{\pi(i)} \geq y_i^t$ for all $i \in I$. Note that the underachievement function $M_{\alpha\rho}(\mathbf{y})$, similar as $\rho(\mathbf{y})$ depends only on the distribution of outcomes. Further, if $\mathbf{y}' \geq \mathbf{y}''$, then $\mathbf{y}' = \mathbf{y}'' + (\mathbf{y}' - \mathbf{y}'')$ and $\mathbf{y}' - \mathbf{y}'' \geq 0$. Hence, due to concavity and positive homogeneity, $M_{\alpha\rho}(\mathbf{y}') \geq M_{\alpha\rho}(\mathbf{y}'') + M_{\alpha\rho}(\mathbf{y}' - \mathbf{y}'')$. Moreover, due to the bound (25), $M_{\alpha\rho}(\mathbf{y}' - \mathbf{y}'') \geq \mu(\mathbf{y}' - \mathbf{y}'') - \Delta(\mathbf{y}' - \mathbf{y}'') \geq \mu(\mathbf{y}' - \mathbf{y}'') - \mu(\mathbf{y}' - \mathbf{y}'') = 0$. Thus, $M_{\alpha\rho}(\mathbf{y})$ satisfies also the requirement of monotonicity. Hence, $M_{\alpha\rho}(\mathbf{y}') \geq M_{\alpha\rho}(\mathbf{y}^t)$. Further, let us notice that $\mathbf{y}^k = \lambda\bar{\mathbf{y}}^{k-1} + (1-\lambda)\mathbf{y}^{k-1}$ where $\bar{\mathbf{y}}^{k-1} = \mathbf{y}^{k-1} - (y_{i'} - y_{i''})\mathbf{e}_{i'} + (y_{i'} - y_{i''})\mathbf{e}_{i''}$ and $\lambda = \varepsilon/(y_{i'} - y_{i''})$. Vector $\bar{\mathbf{y}}^{k-1}$ has the same distribution of coefficients as $\mathbf{y}^{k-1}$ (actually it represents results of swapping $y_{i'}$ and $y_{i''}$). Hence, due to concavity of $M_{\alpha\rho}(\mathbf{y})$, one gets $M_{\alpha\rho}(\mathbf{y}^k) \geq \lambda M_{\alpha\rho}(\bar{\mathbf{y}}^{k-1}) + (1-\lambda) M_{\alpha\rho}(\mathbf{y}^{k-1}) = M_{\alpha\rho}(\mathbf{y}^{k-1})$. Thus, $M_{\alpha\rho}(\mathbf{y}') \geq M_{\alpha\rho}(\mathbf{y}'')$ which justifies the fair α-consistency of $\rho(\mathbf{y})$. ∎

For strong fair α-consistency some strict monotonicity and concavity properties of the underachievement function are needed. Obviously, there does not exist any inequality measure which is positively homogeneous and simultaneously strictly convex. However, one may notice from the proof of Theorem 2 that only convexity properties on equally distributed outcome vectors are important for monotonous underachievement functions.

We say that inequality measure $\rho(\mathbf{y}) \geq 0$ is *strictly convex on equally distributed outcome vectors*, if

$$\rho(\lambda\mathbf{y}' + (1-\lambda)\mathbf{y}'') < \lambda\rho(\mathbf{y}') + (1-\lambda)\rho(\mathbf{y}'')$$

for $0 < \lambda < 1$ and any two vectors $\mathbf{y}' \neq \mathbf{y}''$ representing the same outcomes distribution as some $\mathbf{y}$, i.e., $\mathbf{y}' = (y_{\pi'(1)}, \ldots, y_{\pi'(m)})$ π' and $\mathbf{y}'' = (y_{\pi''(1)}, \ldots, y_{\pi''(m)})$ for some permutations π' and π'', respectively.

Theorem 3. *Let $\rho(\mathbf{y}) \geq 0$ be a convex, positively homogeneous and translation invariant (dispersion type) inequality measure. If $\rho(\mathbf{y})$ is also strictly convex on equally distributed outcomes and $\alpha\rho(\mathbf{y})$ is strictly Δ-bounded, then the measure $\rho(\mathbf{y})$ is fairly strongly α-consistent in the sense of (24).*

Proof. The relation of weak fair dominance $\mathbf{y}' \succeq_e \mathbf{y}''$ denotes that there exists a finite sequence of vectors $\mathbf{y}^0 = \mathbf{y}'', \mathbf{y}^1, \ldots, \mathbf{y}^t$ such that $\mathbf{y}^k = \mathbf{y}^{k-1} - \varepsilon_k \mathbf{e}_{i'} + \varepsilon_k \mathbf{e}_{i''}$, $0 \leq \varepsilon_k \leq y_{i'}^{k-1} - y_{i''}^{k-1}$ for $k = 1, 2, \ldots, t$ and there exists a permutation π such that $y'_{\pi(i)} \geq y_i^t$ for all $i \in I$. The strict fair dominance $\mathbf{y}' \succ_e \mathbf{y}''$ means that $y'_{\pi(i)} > y_i^t$ for some $i \in I$ or at least one ε_k is strictly positive. Note that the underachievement function $M_{\alpha\rho}(\mathbf{y})$ is strictly monotonous and strictly convex on equally distributed outcome vectors. Hence, $M_{\alpha\rho}(\mathbf{y}') > M_{\alpha\rho}(\mathbf{y}'')$ which justifies the fair strong α-consistency of the measure $\rho(\mathbf{y})$. ∎

The specific case of fair 1-consistency is also called *the mean-complementary fair consistency*. Note that the fair $\bar{\alpha}$-consistency of measure $\rho(\mathbf{y})$ actually guarantees the mean-complementary fair consistency of measure $\alpha\rho(\mathbf{y})$ for all $0 < \alpha \leq \bar{\alpha}$, and the same remain valid for the strong consistency properties. It follows from a possible expression of $\mu(\mathbf{y}) - \alpha\rho(\mathbf{y})$ as the convex combination of $\mu(\mathbf{y}) - \bar{\alpha}\rho(\mathbf{y})$ and $\mu(\mathbf{y})$. Hence, for any $\mathbf{y}' \succeq_e \mathbf{y}''$, due to $\mu(\mathbf{y}') \geq \mu(\mathbf{y}'')$ one gets $\mu(\mathbf{y}') - \alpha\rho(\mathbf{y}') \geq \mu(\mathbf{y}'') - \alpha\rho(\mathbf{y}'')$ in the case of the fair $\bar{\alpha}$-consistency of measure $\rho(\mathbf{y})$ (or respective strict inequality in the case of strong consistency). Therefore, while analyzing specific inequality measures we seek the largest values α guaranteeing the corresponding fair consistency.

As mentioned, typical inequality measures are convex and many of them are positively homogeneous. Moreover, the measures such as the mean absolute (downside) semideviation $\bar{\delta}(\mathbf{y})$ (18), the standard downside semideviation $\bar{\sigma}(\mathbf{y})$ (19), and the mean absolute difference $\Gamma(\mathbf{y})$ (12) are Δ-bounded. Indeed, one may easily notice that $\mu(\mathbf{y}) - y_i \leq \bar{\Delta}(\mathbf{y})$ and therefore $\bar{\delta}(\mathbf{y}) \leq \frac{1}{m}\sum_{i\in I}\bar{\Delta}(\mathbf{y}) = \bar{\Delta}(\mathbf{y})$, $\bar{\sigma}(\mathbf{y}) \leq \sqrt{\bar{\Delta}(\mathbf{y})^2} = \bar{\Delta}(\mathbf{y})$ and $\Gamma(\mathbf{y}) = \frac{1}{m^2}\sum_{i\in I}\sum_{j\in I}(\max\{y_i, y_j\} - \mu(\mathbf{y})) \leq \bar{\Delta}(\mathbf{y})$. Actually, all these inequality measures are strictly $\bar{\Delta}$-bounded since for any unequal outcome vector at least one outcome must be below the mean thus leading to strict inequalities in the above bounds. Obviously, $\bar{\Delta}$-bounded (but not strictly) is also the maximum absolute downside deviation $\bar{\Delta}(\mathbf{y})$ itself. This allows us to justify the maximum downside deviation $\bar{\Delta}(\mathbf{y})$ (17), the mean absolute (downside) semideviation $\bar{\delta}(\mathbf{y})$ (18), the standard downside semideviation $\bar{\sigma}(\mathbf{y})$ (19) and the mean absolute difference $\Gamma(\mathbf{y})$ (12) as fairly 1-consistent (mean-complementary fairly consistent) in the sense of (23).

We emphasize that, despite the standard semideviation is a fairly 1-consistent inequality measure, the consistency is not valid for variance, semivariance and even for the standard deviation. These measures, in general, do not satisfy the all assumptions of Theorem 2. Certainly, we have enumerated only the simplest inequality measures studied in the resource allocation context which satisfy the assumptions of Theorem 2 and thereby they are fairly 1-consistent. Theorem 2 allows one to show this property for many other measures. In particular, one may easily find out that any convex combination of fairly α-consistent inequality measures remains also fairly α-consistent. On the other hand, among typical inequality measures the mean absolute difference seems to be the only one meeting the stronger assumptions of Theorem 3 and thereby maintaining the strong consistency.

As mentioned, the mean absolute deviation is twice the mean absolute downside semideviation which means that $\alpha\delta(\mathbf{y})$ is $\bar{\Delta}$-bounded for any $0 < \alpha \leq 0.5$. The symmetry of mean absolute semideviations $\bar{\delta}(\mathbf{y}) = \sum_{i\in I}(y_i - \mu(\mathbf{y}))_+ = \sum_{i\in I}(\mu(\mathbf{y}) - y_i)_+$ can be also used to derive some $\bar{\Delta}$-boundedness relations for other inequality measures. In particular, one may find out that for m-dimensional outcome vectors of unweighted problem, any downside semideviation from the mean cannot be larger than $m-1$ upper semideviations. Hence, the maximum absolute deviation satisfies the inequality $\frac{1}{m-1}R(\mathbf{y}) \leq \bar{\Delta}(\mathbf{y})$, while the maximum absolute difference fulfills $\frac{1}{m}d(\mathbf{y}) \leq \bar{\Delta}(\mathbf{y})$. Similarly, for the standard deviation one gets $\frac{1}{\sqrt{m-1}}\delta(\mathbf{y}) \leq \bar{\Delta}(\mathbf{y})$. Actually, $\alpha\sigma(\mathbf{y})$ is strictly $\bar{\Delta}$-bounded for any $0 < \alpha \leq 1/\sqrt{m-1}$ since for any unequal outcome vector at least one outcome must be below the mean thus leading to strict inequalities in the above bounds. These allow us to justify the mean absolute semideviation with $0 < \alpha \leq 0.5$, the maximum absolute deviation with $0 < \alpha \leq \frac{1}{m-1}$, the maximum absolute difference with $0 < \alpha \leq \frac{1}{m}$ and the standard deviation with $0 < \alpha \leq \frac{1}{\sqrt{m-1}}$ as fairly α-consistent within the specified intervals of α. Moreover, the α-consistency of the standard deviation is strong.

Table 1: Fair consistency results.

Measure			α–consistency	
Mean abs. semidev.	$\bar{\delta}(\mathbf{y})$	(18)	1	
Mean abs. dev.	$\delta(\mathbf{y})$	(14)	0.5	
Max. semidev.	$\bar{\Delta}(\mathbf{y})$	(17)	1	
Max. abs. dev.	$R(\mathbf{y})$	(15)	$\frac{1}{m-1}$	
Mean abs. diff.	$\Gamma(\mathbf{y})$	(12)	1	strong
Max. abs. diff.	$d(\mathbf{y})$	(13)	$\frac{1}{m}$	
Standard semidev.	$\bar{\sigma}(\mathbf{y})$	(19)	1	
Standard dev.	$\sigma(\mathbf{y})$	(16)	$\frac{1}{\sqrt{m-1}}$	strong

The fair consistency results for basic dispersion type inequality measures considered in resource allocation problems are summarized in Table 1 where α values are given and the strong consistency is indicated. Table 1 points out how the inequality measures can be used in resource allocation models to guarantee their harmony both with outcome maximization (Pareto-optimality) and with inequalities minimization (Pigou-Dalton equity theory). Exactly, for each inequality measure applied with the corresponding value α from Table 1 (or smaller positive value), every efficient solution of the bicriteria problem (22), ie. $\max\{(\mu(\mathbf{f}(\mathbf{x})), \mu(\mathbf{f}(\mathbf{x})) - \alpha\rho(\mathbf{f}(\mathbf{x}))) : \mathbf{x} \in Q\}$, is a fairly efficient allocation pattern, except for outcomes with identical values of $\mu(\mathbf{y})$ and $\rho(\mathbf{y})$. In the case of strong consistency (as for mean absolute difference or standard deviation), every solution $\mathbf{x} \in Q$ efficient to (22) is, unconditionally, fairly efficient.

The consistency results summarized in Table 1 are sufficient conditions. This means that whenever the α limit is observed the corresponding consistency relation is valid for any problem. It may happen that for a specific problem instance and a specific inequality measure the fair consistency is valid for larger values of α. Nevertheless, we have provided strict bounds in the sense that for a larger value of α there exists a resource allocation problem on which the fair consistency is not valid, and the bicriteria problem (22) may generate dominated solution.

5 CONCLUSIONS

The problems of efficient and fair resource allocation arise in various systems which serve many users. Fairness is, essentially, an abstract socio-political concept that implies impartiality, justice and equity. Nevertheless, in operations research it was quantified with various solution concepts (Denda et al., 2000). The equitable optimization with the preference structure that complies with both the efficiency (Pareto-optimality) and with the Pigou-Dalton principle of transfers may be used to formalize the fair solution concepts. Multiple criteria models equivalent to equitable optimization allows to generate a variety of fair and efficient resource allocation patterns (Kostreva et al., 2004; Ogryczak et al., 2008).

In this paper we have analyzed how scalar inequality measures can be used to guarantee the fair consistency. It turns out that several inequality measures can be combined with the mean itself into the optimization criteria generalizing the concept of the worst outcome and generating fairly consistent underachievement measures. We have shown that properties of convexity and positive homogeneity together with being bounded by the maximum downside semideviation are sufficient for a typical inequality measure to guarantee the corresponding fair consistency. It allows us to identify various inequality measures which can be effectively used to incorporate fairness factors into various resource allocation problems while preserving the consistency with outcomes maximization. Among others, the standard semideviation and the mean semideviation turn out to be such a consistent inequality measure while the mean absolute difference is strongly consistent.

Our analysis is related to the properties of solutions to resource allocation models. It has been shown how inequality measures can be included into the models avoiding contradiction to the maximization of outcomes. We do not analyze algorithmic issues for the specific resource allocation problems. Generally, the requirement of convexity necessary for the consistency, guarantees that the corresponding optimization criteria belong to the class of convex optimization, not complicating the original resource allocation model with any additional discrete structure. Many of the inequality measures, we analyzed, can be implemented with auxiliary linear programming constraints. Nevertheless, further research on efficient computational algorithms for solving the specific models is necessary.

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