Reference Distribution — An Interactive Approach to Multiple Homogeneous and Anonymous Criteria

Włodzimierz Ogryczak

Mathematics and Informatics, Warsaw University, 02–097 Warsaw, Poland

Abstract. There are several decision problems with multiple homogeneous and anonymous criteria where the preference model needs to satisfy the principle of anonymity (symmetry with respect to permutations of criteria). The standard reference point method cannot be directly applied to such problems. In this paper we develop, as an analogue of the reference point method, the reference distribution method taking into account both the efficiency principle and the principle of anonymity. All the solutions generated during the interactive process belong to the symmetrically efficient set which is a subset of the standard efficient set. It means, the achievement vector of the generated solution is neither dominated by another achievement vector nor by any permutation of some achievement vector.

Key Words. Symmetric Efficiency, Interactive Methods, Reference Point

1 Introduction

Consider a decision problem defined as an optimization problem with m homogeneous objective functions. For simplification we assume, without loss of generality, that the objective functions are to be minimized. The problem can be formulated as follows

$$\min \{\mathbf{F}(\mathbf{x}) \ : \quad \mathbf{x} \in Q\} \tag{1}$$

where

$\mathbf{F} = (f_1, \ldots, f_m)$	is a vector–function that maps the decision space $X = R^n$ into the criterion space $Y = R^m$,
$Q \subset X$	denotes the feasible set,
$\mathbf{x} \in X$	denotes the vector of decision variables.

The elements of the criterion space we refer to as achievement vectors. An achievement vector $\mathbf{y} \in Y$ is attainable if it expresses outcomes of a feasible solution $\mathbf{x} \in Q$ ($\mathbf{y} = \mathbf{F}(\mathbf{x})$). The set of all the attainable achievement vectors is denoted by Y_a, i.e. $Y_a = \{\mathbf{y} \in Y \ : \ \mathbf{y} = \mathbf{F}(\mathbf{x}), \ \mathbf{x} \in Q\}$.

Model (1) only specifies that we are interested in minimization of all objective functions f_i for $i \in I = \{1, 2, \ldots, m\}$. In order to make it operational, one needs to assume some solution concept specifying what it means to minimize multiple objective functions. The solution concepts are defined by properties of the corresponding preference model. We assume that solution concepts depend only on evaluation of the achievement vectors do not taking into account other solution properties not represented within achievement vectors. Thus, we can limit our considerations to the preference model in the criterion space Y.

The preference model is completely characterized by the relation of weak preference (c.f., [5]), denoted hereafter with $\preceq$. Namely, we say that achievement vector $\mathbf{y}' \in Y$ is (strictly) preferred to $\mathbf{y}'' \in Y$ ($\mathbf{y}' \prec \mathbf{y}''$) iff $\mathbf{y}' \preceq \mathbf{y}''$ and $\mathbf{y}'' \not\preceq \mathbf{y}'$. Similarly, we say that achievement vector $\mathbf{y}' \in Y$ is indifferent or equally preferred to $\mathbf{y}'' \in Y$ ($\mathbf{y}' \cong \mathbf{y}''$) iff $\mathbf{y}' \preceq \mathbf{y}''$ and $\mathbf{y}'' \preceq \mathbf{y}'$.

The standard preference model related to the Pareto–optimal solution concept assumes that the preference relation $\preceq$ is reflexive

$$\mathbf{y} \preceq \mathbf{y} \tag{2}$$

transitive

$$(\mathbf{y}' \preceq \mathbf{y}'' \quad \text{and} \quad \mathbf{y}'' \preceq \mathbf{y}''') \quad \Rightarrow \quad \mathbf{y}' \preceq \mathbf{y}''' \tag{3}$$

and strictly monotonic

$$\mathbf{y} - \varepsilon \mathbf{e}_i \prec \mathbf{y} \quad \text{for} \quad \varepsilon > 0 \tag{4}$$

where $\mathbf{e}_i$ denotes the i–th unit vector in the criterion space. The last assumption expresses that for each individual objective function less means better (minimization).

We focus on multiple criteria problems with homogeneous and anonymous objective functions. Therefore, we assume also that the preference relation $\preceq$ is anonymous (or impartial), i.e.

$$(y_{\tau(1)}, y_{\tau(2)}, \ldots, y_{\tau(m)}) \cong (y_1, y_2, \ldots, y_m) \tag{5}$$

for any permutation τ of I.

There are several decision problems with multiple homogeneous and anonymous criteria. As an example one may consider location problems. The generic location problem may be stated as follows. There is given a set of m clients (spatial units). There is also given a set of n potential locations for the facilities. It may be in particular a subset (or the entire set) of points representing the clients. Further, the number p of facilities to be located is given ($p \leq n$). The main decisions to be made in the location problem can be described with the binary variables x_j ($j = 1, 2, \ldots, n$) equal to 1 if location j is to be used and equal to 0 otherwise. To meet the problem requirements the decision variables x_j have to satisfy the constraint $\sum_{j=1}^{n} x_j = p$. Further, let us assume that for each client $i = 1, 2, \ldots, m$ there is defined a function

$f_i(\mathbf{x})$ of the location pattern $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. The function measures quality of the location pattern with respect to the satisfaction of client i. In typical formulations of location problems this function is usually related to the distances and thereby its less value means higher service quality and client satisfaction. Therefore, each function f_i needs to be minimized. Thus the generic location problem can be viewed as the following multiple criteria minimization problem

$$\min \{\mathbf{F}(\mathbf{x}) \ : \ \sum_{j=1}^{n} x_j = p, \quad x_j \in \{0,1\} \quad \text{for} \quad j = 1, 2, \ldots, n\} \tag{6}$$

The individual objective functions f_i are usually conflicting when minimized. Therefore, (6) can be considered a multiple criteria decision problem with homogeneous objective functions. Moreover, while locating public facilities, the distribution of distances among the clients is the crucial issue and the preference model should satisfy the property of anonymity.

2 Symmetric efficiency

It is clear, or rather commonly accepted, that an achievement vector is better than another if all its individual achievements are better or at least one individual achievement is better whereas no other one is worse. In fact, it is the most general assumption about the preference model underlying the multiple criteria optimization. This assumption is equivalent to properties (2)–(4) of the preference model. It is mathematically formalized with the domination relation defined on the criterion space Y.

Definition 1 *We say that achievement vector* $\mathbf{y}' \in Y$ *dominates* $\mathbf{y}'' \in Y$, *or* $\mathbf{y}''$ *is dominated by* $\mathbf{y}'$, *if* $y'_i \leq y''_i$ *for all* $i \in I$ *and for at least one index* i_0 *strict inequality holds* $(y'_{i_0} < y''_{i_0})$.

Unfortunately, there usually does not exist an attainable achievement vector that dominates all the others with respect to all the criteria. Thus, in terms of the domination relation, we cannot distinguish the best attainable achievement vector. We can only distinguish the attainable achievement vectors which are not dominated by the others.

Definition 2 *We say that achievement vector* $\mathbf{y} \in Y_a$ *is nondominated, if does not exist* $\mathbf{y}' \in Y_a$ *such that* $\mathbf{y}'$ *dominates* $\mathbf{y}$.

Definition 3 *We say that feasible solution* $\mathbf{x} \in Q$ *is an efficient (Pareto-optimal) solution of the multiple criteria problem (1), if* $\mathbf{y} = \mathbf{F}(\mathbf{x})$ *is a nondominated achievement vector.*

In our problem all the functions are equally important and the preference model satisfies the property of anonymity (5). That means we are interested in comparison rather sets of outcomes than achievement vectors. Therefore, for the problems with homogeneous and equally important objective functions we should introduce an efficiency concept based rather on the set of outcomes than on the achievement vectors. It means, we need to consider the symmetric domination relation which is not affected by any permutation of the achievement vector coefficients.

Definition 4 *We say that achievement vector* $\mathbf{y}' \in Y$ *symmetrically dominates* $\mathbf{y}'' \in Y$*, or* $\mathbf{y}''$ *is symmetrically dominated by* $\mathbf{y}'$*, if there exist permutations* τ' *and* τ'' *such that* $y'_{\tau'(i)} \leq y''_{\tau''(i)}$ *for all* $i \in I$ *and for at least one index* i_0 *strict inequality holds* ($y'_{\tau'(i_0)} < y''_{\tau''(i_0)}$).

Definition 5 *We say that feasible solution* $\mathbf{x} \in Q$ *is a symmetrically efficient solution of the multiple criteria problem (1), if* $\mathbf{y} = \mathbf{F}(\mathbf{x})$ *is symmetrically nondominated.*

The symmetric efficiency is stronger than the standard efficiency and the symmetrically efficient set is a subset of the standard efficient set. The relation of symmetric domination can be expressed as domination of the achievement vectors with coefficients ordered in the weakly decreasing order. This can be mathematically formalized with the ordering map $\Theta : R^m \rightarrow R^m$ such that $\Theta(y_1, y_2, \ldots, y_m) = (\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_m)$, where $\bar{y}_1 \geq \bar{y}_2 \geq \ldots \geq \bar{y}_m$ and $\bar{y}_i = y_{\tau(i)}$ for $i = 1, 2, \ldots, m$ for some permutation τ of I.

Definition 6 *We say that achievement vector* $\mathbf{y}' \in Y$ *dominates* $\mathbf{y}'' \in Y$ *in the ordered sense, or* $\mathbf{y}''$ *is dominated by* $\mathbf{y}'$ *in the ordered sense, if* $\bar{\mathbf{y}}' = \Theta(\mathbf{y}')$ *dominates* $\bar{\mathbf{y}}'' = \Theta(\mathbf{y}'')$*, i.e.* $\bar{y}'_i \leq \bar{y}''_i$ *for all* $i \in I$ *and for at least one index* i_0 *strict inequality holds* ($\bar{y}'_{i_0} < \bar{y}''_{i_0}$).

Proposition 1 *Achievement vector* $\mathbf{y}' \in Y$ *symmetrically dominates* $\mathbf{y}'' \in Y$ *if and only if* $\mathbf{y}'$ *dominates* $\mathbf{y}''$ *in the ordered sense.*

Proof. If $\mathbf{y}'$ dominates $\mathbf{y}''$ in the ordered sense then, obviously, it does also symmetrically dominate. Thus we only need to prove that the symmetric dominance implies the dominance in the ordered sense.

Suppose that achievement vector $\mathbf{y}' \in Y$ symmetrically dominates $\mathbf{y}'' \in Y$. It means, $\mathbf{y}' \neq \mathbf{y}''$ and there exist permutations τ' and τ'' such that the dominance inequalities are valid

$$y'_{\tau'(i)} \leq y''_{\tau''(i)} \quad \text{for} \quad i = 1, 2, \ldots, m$$

We will show that the dominance inequalities remain satisfied if one replaces permutations τ' and τ'' with permutations $\bar{\tau}'$ and $\bar{\tau}''$ sorting the corresponding achievement vectors in the weakly decreasing order. Note that any vector can be sorted by a finite number of comparisons and swappings (if necessary) made on pairs of its coefficients. Suppose that for some pair of indices $i < j$ the corresponding coefficients of one of the achievement vectors, let say $\mathbf{y}'$, violate the weakly decreasing order, i.e., $y'_{\tau'(i)} < y'_{\tau'(j)}$. If, simultaneously, $y''_{\tau''(i)} < y''_{\tau''(j)}$, then one can simply swap the corresponding coefficients of both the achievements vectors preserving the dominance inequalities. If $y''_{\tau''(i)} \geq y''_{\tau''(j)}$, then $y'_{\tau'(i)} < y'_{\tau'(j)} \leq y''_{\tau''(j)} \leq y''_{\tau''(i)}$ and therefore, one can swap the coefficients of vector $\mathbf{y}'$ preserving the dominance inequalities and the weakly decreasing order of the corresponding coefficients in the second achievement vector.

By applying this approach to the both achievement vectors, one finally gets (in a finite number of swappings) the ordered achievement vectors satisfying the dominance inequalities. □

Corollary 1 *Feasible solution $\mathbf{x} \in Q$ is a symmetrically efficient solution of the multiple criteria problem (1), if and only if it is an efficient solution of the ordered multiple criteria problem*

$$\min \{\Theta(\mathbf{F}(\mathbf{x})) \;:\; \mathbf{x} \in Q\} \tag{7}$$

There exist usually many symmetrically nondominated achievement vectors and they are incomparable each other on the basis of the specified set of objective functions. Therefore, there exist usually many symmetrically efficient solutions and they are different not only in the decision space but also in the criterion space. So, there arises a need for further analysis, or rather decision support, to help the decision maker (DM) in selection of one solution for implementation. Of course, the original objective functions do not allow one to select any symmetrically efficient solution as better than any other one. Therefore, this analysis depends usually on additional information about the DM's preferences. The DM, working interactively with a decision support system (DSS), specifies his/her preferences in terms of some control parameters and the DSS provides the DM with a symmetrically efficient solution which is the best according to the specified control parameters. For such an analysis, there is no need to identify the entire symmetrically efficient set prior to the analysis, as contemporary optimization software is powerful enough to be used on–line for direct computations at each interactive step. Thus the DSS can generate at each interactive step only one symmetrically efficient solution that meets the current preferences. Such a DSS can be used for analysis of decision problems with finite as well as infinite efficient sets. There is important, however, that the control parameters provide the completeness of the control (c.f., [7]), i.e., that varying the control parameters the DM can identify every symmetrically nondominated achievement vector.

For an interactive DSS dealing with multiple and homogeneous criteria we need parametric solution concepts generating symmetrically efficient solutions. In the case of the standard efficiency one may consider weighting of objective functions. In the case of anonymous criteria we cannot assign various weights to individual objective functions. Due to Corollary 1, the weights should be assigned rather to the specific coefficients of the ordered achievement vectors. Such an ordered weighting approach was proposed by Yager [8] in the so–called Ordered Weighted Averaging (OWA) aggregation. Applying the OWA aggregation operator to the multiple criteria problem (1) we get the following single objective problem

$$\min \{\sum_{i=1}^{m} w_i \bar{y}_i \ : \quad \bar{\mathbf{y}} = \Theta(\mathbf{F}(\mathbf{x})), \quad \mathbf{x} \in Q\} \tag{8}$$

Due to Corollary 1, the following proposition is valid.

Proposition 2 *For any positive weights w_i, any optimal solution to problem (8) is a symmetrically efficient solution of the multiple criteria problem (1).*

Unfortunately, the ordered weighting does not provide us with a complete parameterization of the entire symmetrically efficient set. It is due to the specificity of the linear weighting approach to multiple criteria. In the case when the multiple criteria problem is a discrete one (like the location problem (6)), there exist symmetrically efficient solutions that cannot be generated as optimal solutions to problem (8) with any set of positive weights. We illustrate this with a small example.

Example 1 Let us consider a simple single facility location problem with two clients (C1 and C2) and three potential locations (P1, P2 and P3). The distances between several clients and potential locations are given as follows: $d_{11} = 15$, $d_{12} = 14$, $d_{13} = 12$, $d_{21} = 10$, $d_{22} = 11$, $d_{23} = 12$.

Note that all three feasible solutions are efficient in the standard and symmetric sense. One can easily verify that while dealing with ordered weighting approach, location P2 cannot be selected for any set of positive weights. If $3w_1 < 2w_2$, then location P1 is a unique optimal solution to the problem (8). If $3w_1 > 2w_2$, then location P3 is a unique optimal solution to the problem (8). Finally, if $3w_1 = 2w_2$, then both locations P1 and P3 are optimal. Location P2 is never an optimal solution to the corresponding problem (8). □

In the case of discrete (or more general nonconvex) feasible sets, the entire efficient set can be parameterized with augmented weighted Tchebychev distance function (c.f., [4]). It is used as the basis of the reference point method [6]. Due to Corollary 1, we can apply the reference point method to the ordered problem (7) to parameterize the entire symmetrically efficient set of the original multiple criteria problem (1). In the next section we describe this approach in details.

3 Reference distribution approach

The reference point method [6] is an interactive technique for an open search for a satisficing efficient solution. The basic concept of the interactive scheme is as follows. The DM specifies requirements in terms of aspiration levels for individual objective functions. Depending on the specified aspiration levels a special scalarizing achievement function is built which when minimized generates an efficient solution to the problem. The computed efficient solution is presented to the DM as the current solution allowing comparison with previous solutions and modifications of the aspiration levels if necessary.

The scalarizing achievement function not only guarantees efficiency of the solution but also reflects the DM's expectation as specified via the aspiration levels. In building the function the following assumption regarding the DM's expectations is made: the DM prefers outcomes that satisfy all the aspiration levels to any outcome that does not reach one or more of the aspiration levels. One of the simplest scalarizing functions takes the following form (c.f., [4]):

$$s(\mathbf{y}) = \max_{i=1,\ldots,m} \{\lambda_i(y_i - a_i)\} + \varepsilon \sum_{i=1}^{m} \lambda_i(y_i - a_i) \tag{9}$$

where

- $\mathbf{a}$ denotes the vector of aspiration levels,
- λ is a scaling vector, $\lambda_i > 0$,
- ε is an arbitrarily small positive number.

Minimization of the scalarizing achievement function (9) over the feasible set generates an efficient solution. The selection of the solution within the efficient set depends on two vector parameters: an aspiration vector $\mathbf{a}$ and a scaling vector λ. In practical implementations the former is usually designated as a control tool for direct use by the DM during the interactive analysis. The latter is automatically calculated on the basis of some predecision analysis or adjusted during the interactive process depending on values of the reservation levels used as additional control parameters. The small scalar ε is introduced only to guarantee efficiency in the case of a nonunique optimal solution. It can be replaced by two level lexicographic minimization of the corresponding terms [3]. The reference point approach was successfully implemented in many DSS (c.f., [1]) with real–life applications including multiple criteria location decision problems (see, for example, [2]).

In order to parameterize the entire symmetrically efficient set, one may use the scalarizing achievement function

$$\bar{s}(\mathbf{y}) = \max_{i=1,\ldots,m} \{\lambda_i(\bar{y}_i - \bar{a}_i)\} + \varepsilon \sum_{i=1}^{m} \lambda_i(\bar{y}_i - \bar{a}_i), \quad \bar{\mathbf{y}} = \Theta(\mathbf{y}),\ \bar{\mathbf{a}} = \Theta(\mathbf{a}) \tag{10}$$

where ε is an arbitrarily small positive parameter. Applying function (10) to the multiple criteria problem (1) we get the following parameterized single objective problem generating symmetrically efficient solutions

$$\min \{\bar{s}(\mathbf{y}) \ : \quad \mathbf{y} = \mathbf{F}(\mathbf{x}), \quad \mathbf{x} \in Q\} \tag{11}$$

Parametric problem (11) provides us with a complete parameterization for the symmetrically efficient set of the multiple criteria problem (1). That means, any optimal solution to problem (11) is a symmetrically efficient solution of (1) and any symmetrically efficient solution of the multiple criteria problem (1) can be generated as an optimal solution to problem (11) for some aspiration vector $\mathbf{a}$.

Ordering operator Θ used in the definition of scalarizing achievement function (10), in general, makes the scalarized problem (11) very difficult to implement. Note that even unweighted scalarizing achievement function (10) with all $\lambda_i = 1$ provides us with a complete parameterization of the entire symmetrically efficient set. If we decide to use such unweighted scalarizing achievement function we can form the corresponding scalarized problem (11) without the ordering operator in the following form

$$\begin{aligned}
\text{minimize} \quad & \max_{i=1,\ldots,m} z_i + \varepsilon \sum_{i=1}^{m} z_i \\
\text{subject to} \quad & \mathbf{x} \in Q \\
& z_i = f_i(\mathbf{x}) - \sum_{l=1}^{m} a_l u_{il}, \quad \sum_{l=1}^{m} u_{il} = 1 \quad \text{for} \quad i = 1,2,\ldots,m \\
& \sum_{i=1}^{m} u_{il} = 1 \quad \text{for} \quad l = 1,2,\ldots,m \\
& u_{il} \in \{0,1\} \quad \text{for} \quad i = 1,2,\ldots,m; l = 1,2,\ldots,m
\end{aligned}$$

Note that aspiration vector $\mathbf{a}$ is used in scalarizing achievement function (10) only in its ordered form $\bar{\mathbf{a}}$. Thus it is rather an aspiration set of outcomes than a vector. For problems with large number of objectives, like large location problem (6), we can consider it as an aspiration distribution of outcomes. In fact, for discrete problems with multiple homogeneous criteria we can directly deal with the distribution of outcomes. Let $V = \{v_1, v_2, \ldots, v_r\}$ ($v_1 > \cdots > v_r$) denote the set of all possible values of objective functions f_i for $\mathbf{x} \in Q$. We can introduce then integer functions $h_k(\mathbf{x})$ ($k = 1,2,\ldots,r$) expressing the number of values v_k taken in the achievement vector $\mathbf{F}(\mathbf{x})$. Analytically, functions h_k can be introduced into the model by auxiliary assignment (binary) variables u_{ik} with the following formulas

$$h_k(\mathbf{x}) = \sum_{i=1}^{m} u_{ik} \qquad \text{for} \quad k = 1,2,\ldots,r \tag{12}$$

$$f_i(\mathbf{x}) = \sum_{k=1}^{r} v_k u_{ik}, \quad \sum_{k=1}^{r} u_{ik} = 1 \qquad \text{for} \quad i = 1,2,\ldots,m \tag{13}$$

$$u_{ik} \in \{0,1\} \quad \text{for} \quad i = 1,2,\ldots,m; k = 1,2,\ldots,r \tag{14}$$

Note that in many discrete problems functions h_k can be introduced directly to the model without auxiliary variables u_{ik}. It is possible, in particular, for the location problem with explicit allocation variables.

Having defined functions h_k we can introduce cumulative distribution functions

$$\bar{h}_k(\mathbf{x}) = \sum_{l=1}^{k} h_l(\mathbf{x}) \quad \text{for} \quad k = 1, 2, \ldots, r \tag{15}$$

and consider the corresponding multiple criteria problem

$$\min \{(\bar{h}_1(\mathbf{x}), \bar{h}_2(\mathbf{x}), \ldots, \bar{h}_r(\mathbf{x})) \;:\; \mathbf{x} \in Q\} \tag{16}$$

Proposition 3 *Feasible solution $\mathbf{x} \in Q$ is a symmetrically efficient solution of the multiple criteria problem (1), if and only if it is an efficient solution of the multiple criteria problem (16).*

Proof. Let $\mathbf{x} \in Q$ be a symmetrically efficient solution of problem (1). Suppose that $\mathbf{x}$ is not efficient solution of the distribution problem (16). It means, there exists $\mathbf{x}^0 \in Q$ such that $\bar{h}_k(\mathbf{x}^0) \leq \bar{h}_k(\mathbf{x})$ for $k = 1, 2, \ldots, r$ where for at least one index k_0 strict inequality holds ($\bar{h}_{k_0}(\mathbf{x}^0) < \bar{h}_{k_0}(\mathbf{x})$). Then, obviously, $\Theta(\mathbf{F}(\mathbf{x}^0))$ dominates $\Theta(\mathbf{F}(\mathbf{x}))$ which contradicts symmetric efficiency of $\mathbf{x}$ for problem (1). Thus, symmetric efficiency of vector $\mathbf{x} \in Q$ for problem (1) implies its efficiency for problem (16).

Now, let $\mathbf{x} \in Q$ be an efficient solution of problem (16). Suppose that $\mathbf{x}$ is not symmetrically efficient solution of problem (1). It means, there exists $\mathbf{x}^0 \in Q$ such that $\bar{\mathbf{y}}^0 = \Theta(\mathbf{F}(\mathbf{x}^0))$ dominates $\bar{\mathbf{y}} = \Theta(\mathbf{F}(\mathbf{x}))$. Note that $\bar{h}_k(\mathbf{x}^0) = \bar{h}_k(\mathbf{x}) = 0$ if $v_k > \bar{y}_1^0$ and $v_k > \bar{y}_1$ as well as $\bar{h}_k(\mathbf{x}^0) = \bar{h}_k(\mathbf{x}) = m$ if $v_k < \bar{y}_m^0$ and $v_k < \bar{y}_m$. Moreover, for any $i \in I$ $\bar{y}_i^0 = v_{k'} \leq \bar{y}_i = v_{k''}$ implies $\bar{h}_k(\mathbf{x}^0) \leq \bar{h}_k(\mathbf{x})$ for $k' \leq k \leq k''$. So, achievement vector $\bar{H}(\mathbf{x}^0)$ dominates (in the standard sense) achievement vector $\bar{H}(\mathbf{x})$ which contradicts efficiency of $\mathbf{x}$ for problem (16). Thus, efficiency of vector $\mathbf{x} \in Q$ for problem (16) implies its symmetric efficiency for problem (1). □

Due to Proposition 3 we can apply the standard reference point method to the distribution multiple criteria problem (16) for an interactive analysis of the symmetrically efficient set of problem (1). The corresponding scalarizing achievement function takes then the following form

$$s(\mathbf{x}) = \max_{k=1,\ldots,r} \{\lambda_k(\bar{h}_k(\mathbf{x}) - \bar{q}_k)\} + \varepsilon \sum_{k=1}^{r} \lambda_k(\bar{h}_k(\mathbf{x}) - \bar{q}_k) \tag{17}$$

where

- $\bar{\mathbf{q}}$ denotes the vector of aspiration levels for the cumulative distribution of outcomes,
- λ is a scaling vector, $\lambda_k > 0$,
- ε is an arbitrarily small positive number.

Aspiration distribution vector $\bar{\mathbf{q}}$ is the main control tool for direct use by the DM during an interactive analysis. Scaling factors λ_k can be used as auxiliary control parameters and modified by the DM during the interactive process. Note that, in the case of large r, the DM does not need to deal with all the aspiration coefficients $\bar{q}_k$. As $\bar{\mathbf{q}}$ represents the reference cumulative distribution, it can be specified with only a few coefficients $\bar{q}_k$ and automatic interpolation of values for the remaining coefficients.

4 Concluding remarks

The reference point method is a very convenient technique for interactive analysis of the multiple criteria optimization problems. It provides the DM with a tool for an open analysis of the efficient frontier. The interactive analysis is navigated with the commonly accepted control parameters expressing aspiration levels for the individual objective functions.

There are several decision problems with multiple homogeneous and anonymous criteria where the preference model needs to satisfy the principle of anonymity (symmetry with respect to permutations of criteria). The standard reference point method cannot be directly applied to such problems. In this paper we have developed, as an analogue of the reference point method, the reference distribution method taking into account both the efficiency principle and the principle of anonymity. All the solutions generated during the interactive process belong to the symmetrically efficient set. The interactive analysis of the symmetrically efficient set is controlled with the aspiration cumulative distribution of outcomes.

References

[1] Lewandowski, A., A.P. Wierzbicki (eds.): *Aspiration Based Decision Support Systems — Theory, Software and Applications*. Springer–Verlag, Berlin 1989

[2] Malczewski, J., W. Ogryczak: An interactive approach to the central facility location problem: Locating pediatric hospitals in Warsaw. Geographical Analysis 22 (1990) 244–258

[3] Ogryczak, W.: A goal programming model of the reference point method. Annals of Operations Research 51 (1994) 33–44

[4] Steuer, R.E.: *Multiple Criteria Optimization — Theory, Computation & Applications*. John Wiley, New York 1986

[5] Vincke, Ph.: *Multicriteria Decision–Aid*. John Wiley, New York 1992

[6] Wierzbicki, A.P.: A mathematical basis for satisficing decision making. Mathematical Modelling 3 (1982) 391–405

[7] Wierzbicki, A.P.: On completeness and constructiveness of parametric characterizations to vector optimization problems. OR Spektrum 8 (1986) 73–87

[8] Yager, R.R.: On ordered weighted averaging aggregation operators in multicriteria decision making. IEEE Transactions on Systems, Man and Cybernetics 18 (1988) 183–190