Institute of Control and Computation Engineering
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Stability analysis of nonlinear control systems with fuzzy DMC controllers

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SCHEME OF PRESENTATION:

Introduction

DMC algorithm (C.R. Cutler and B.L. Ramaker, 1979):

- a long–range horizon predictive control algorithm;
- many advantages – possibility of taking into account:
  - constraints,
  - future set–point changes,
  - anticipated disturbance changes.

FDMC controllers:

- combination of two ideas:
  - long–range horizon DMC predictive controller,
  - fuzzy Takagi–Sugeno (multiregional) approach;
- the advantages of both techniques are included

Predictive control

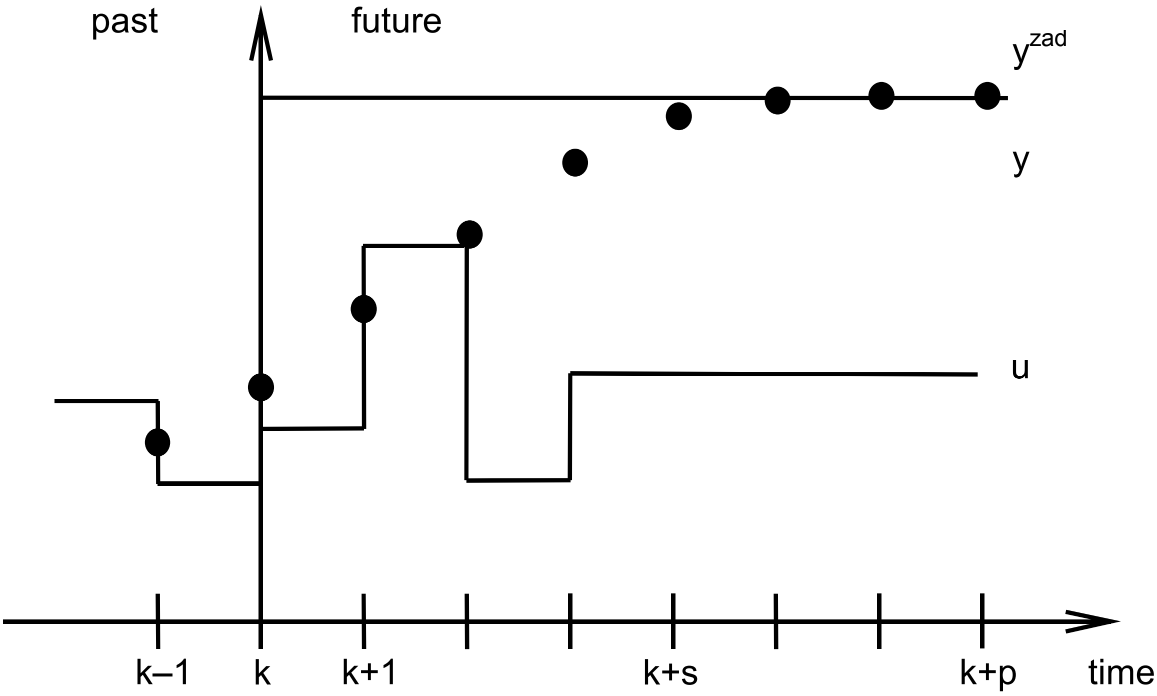


Fig. 1. Illustration of the prediction horizon (p) and control horizon (s) situation in present time step k; u – manipulated variable, y – predicted output, y^{zad} – set–point value

Conventional DMC algorithm based on analytical formulation

The basic idea of the DMC algorithm:

$$\min_{\Delta u} \sum_{i=1}^{p} \left(y_k^{sp} - y_{k+i}^{pred} \right)^2 + \lambda \cdot \sum_{i=0}^{s-1} \left(\Delta u_{k+i} \right)^2$$

$$y_{k+i}^{pred} = y_k + w_{k+i} + \Delta y_{k+i}, \; i = 1, \ldots, p,$$

A unique solution:

$$\boldsymbol{\Delta u}=(\boldsymbol{A}^T\cdot\boldsymbol{A}+\lambda\cdot\boldsymbol{I})^{-1}\cdot\boldsymbol{A}^T\cdot(\boldsymbol{e}-\boldsymbol{w}),$$

where

$$\Delta u=\left[\Delta u_k,\ldots,\Delta u_{k+s-1}\right]^T \text{ – future control increments,}$$

$$w=\left[w_{k+1},\ldots,w_{k+p}\right]^T \text{ – depends on the past,}$$

$$e=\left[y_k^{sp}-y_k,\ldots,y_k^{sp}-y_k\right]^T$$

Only the first element of $\boldsymbol{\Delta u}$ is used at each time step.

The structure of the controller

$$u_k=u_{k-1}+r_0\cdot e_k+\sum_{j=1}^{p-1}r_j\cdot\Delta u_{k-j}$$

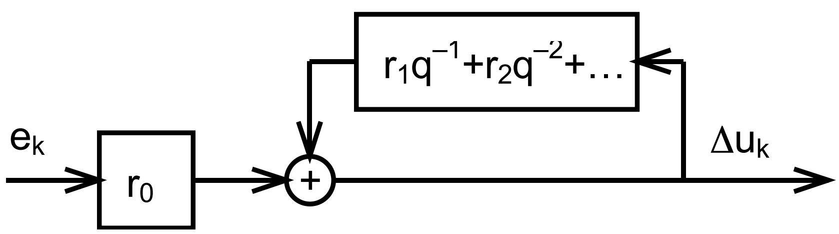


Fig. 2. Block diagram of the DMC controller

Fuzzy DMC algorithm based on analytical formulation

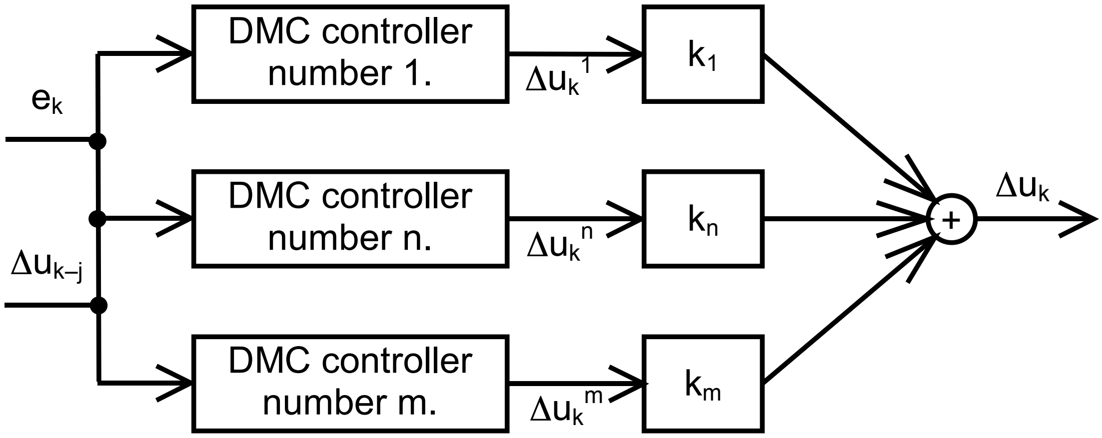


Fig. 3. Block diagram of the FDMC1 controller

— The FDMC1 controller is a combination of many sub–controllers.

— Parameters of sub–controllers are derived once, off–line.

— It is enough to sum up weighted outputs of those controllers in order to calculate the output value of the whole controller.

Control plant

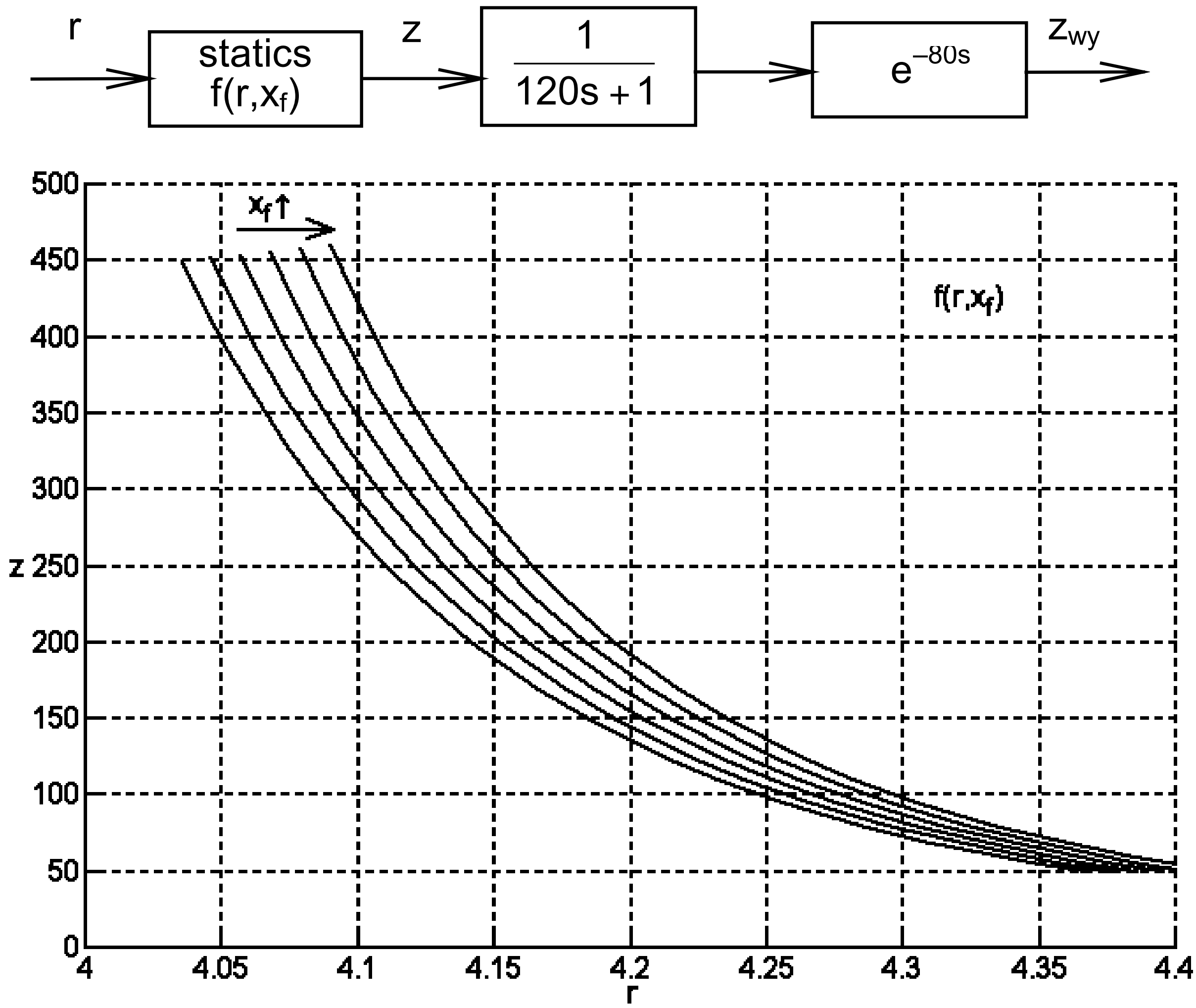


Fig. 4. Block diagram and static characteristics of the control plant

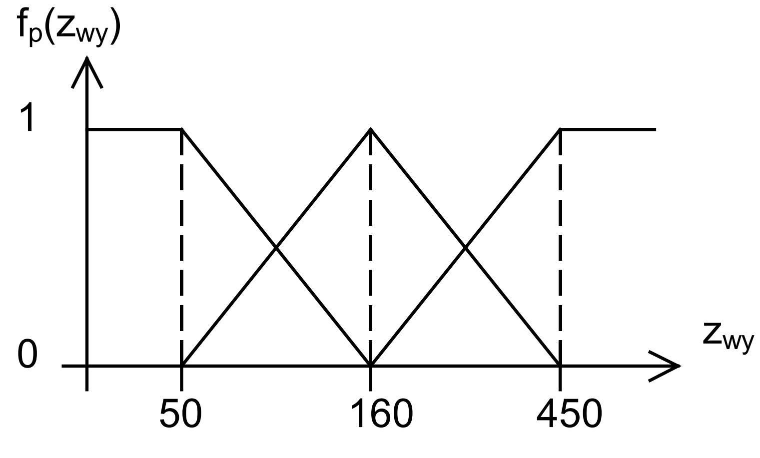
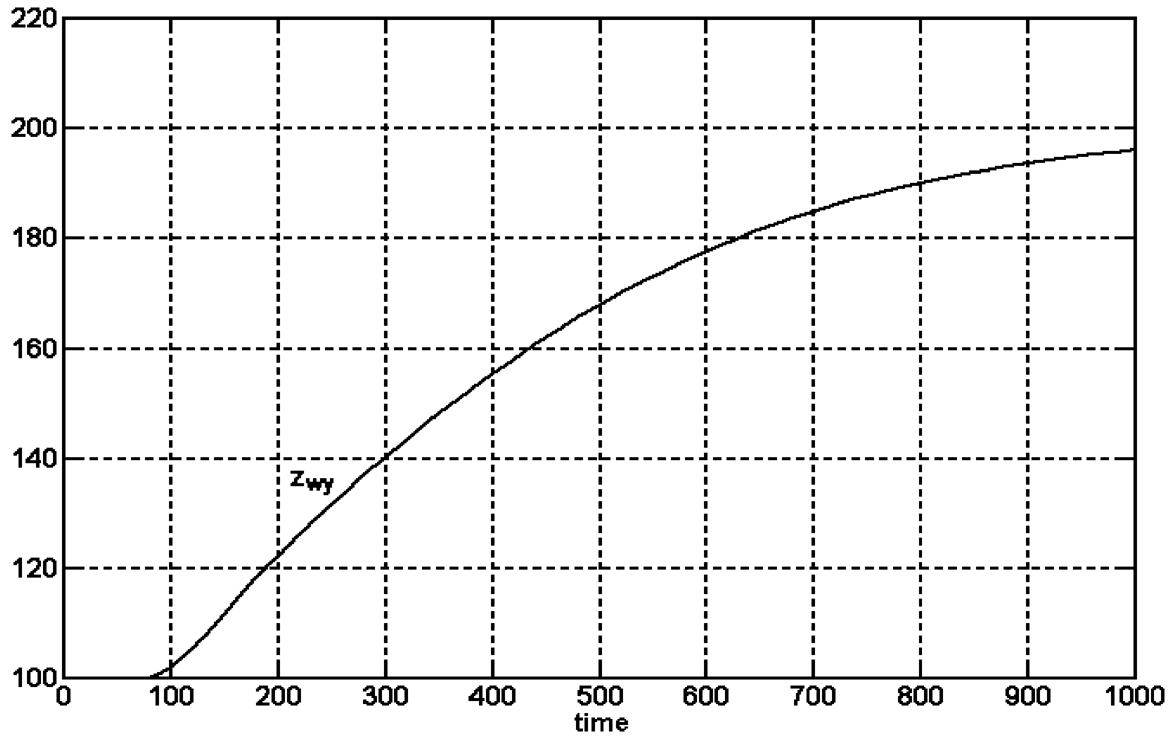


Fig. 5. Membership functions

a)



b)

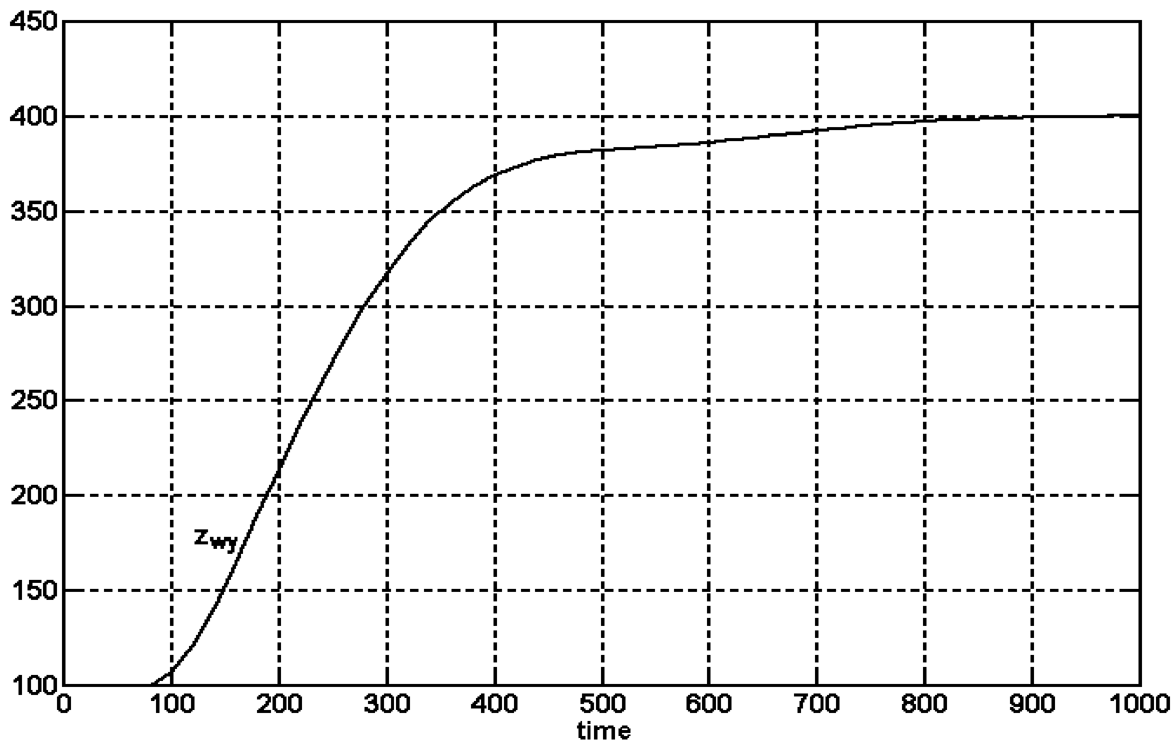
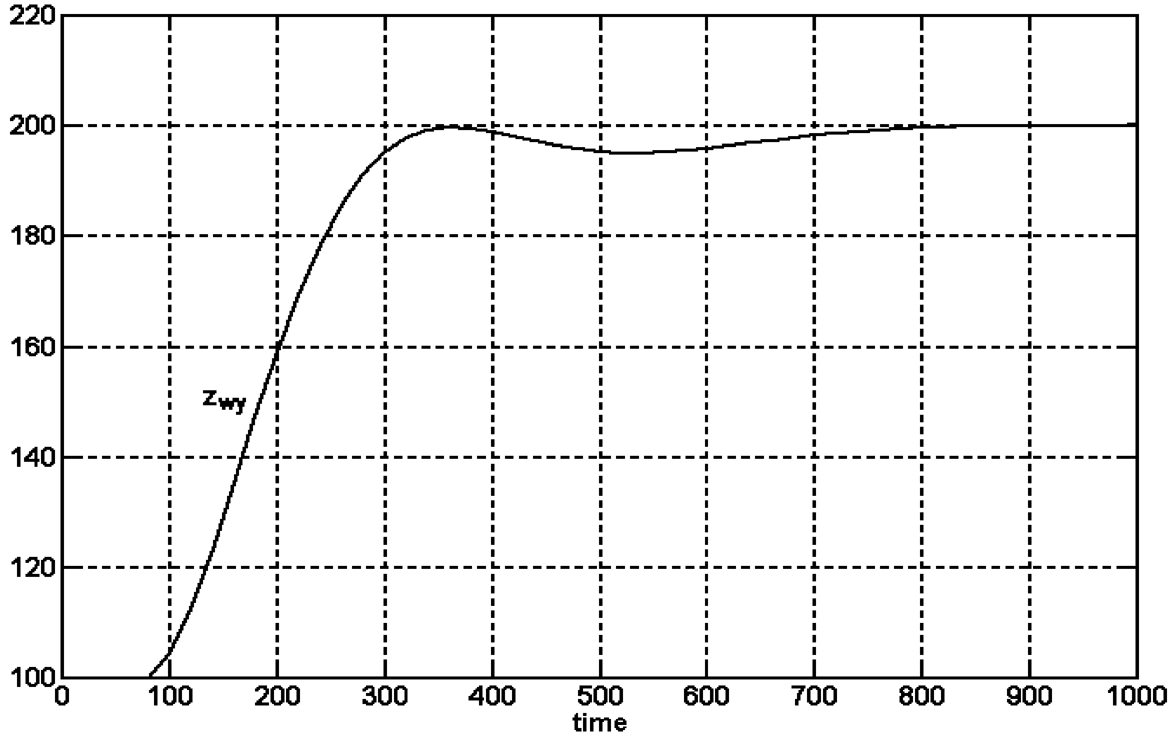


Fig. 6. Responses with “normal” DMC controller designed for set–point 400 ppm; set–point change from z_0 = 100 ppm to a) z_{zad} = 200 ppm; b) z_{zad} = 400 ppm;

a)



b)

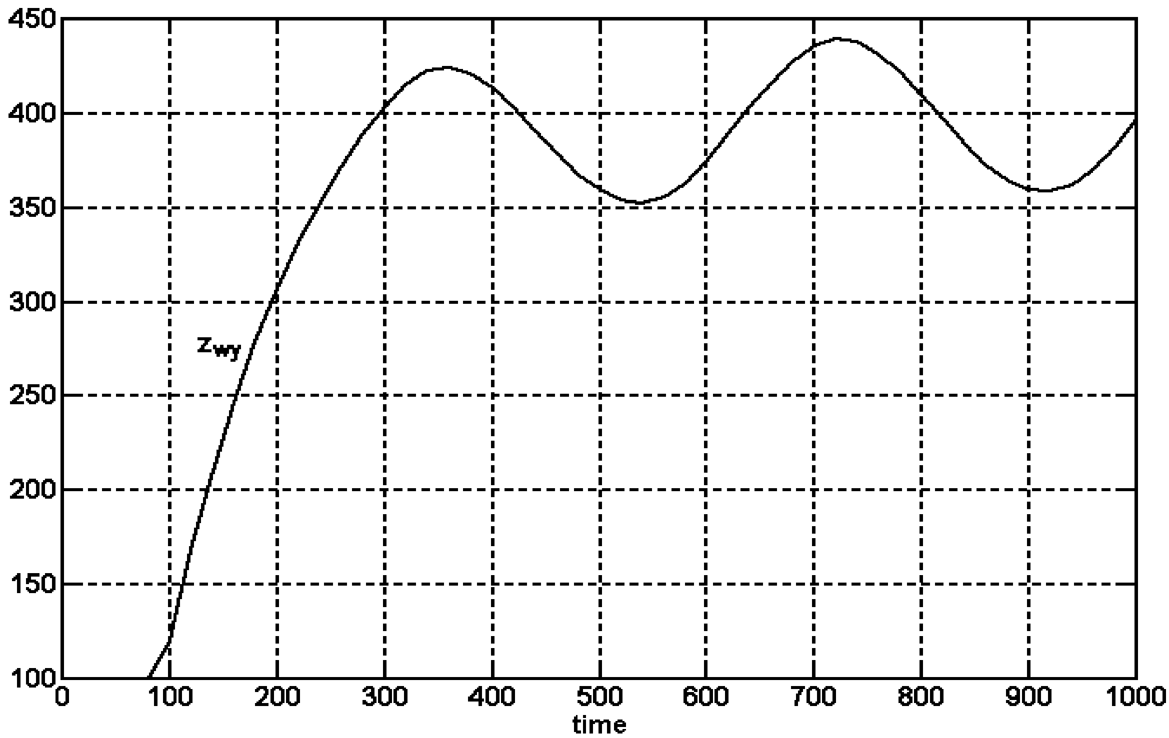


Fig. 7. Responses with "normal" DMC controller designed for set–point 200 ppm; set–point change from $z_0 = 100$ ppm to a) $z_{zad} = 200$ ppm; b) $z_{zad} = 400$ ppm;

a) b)

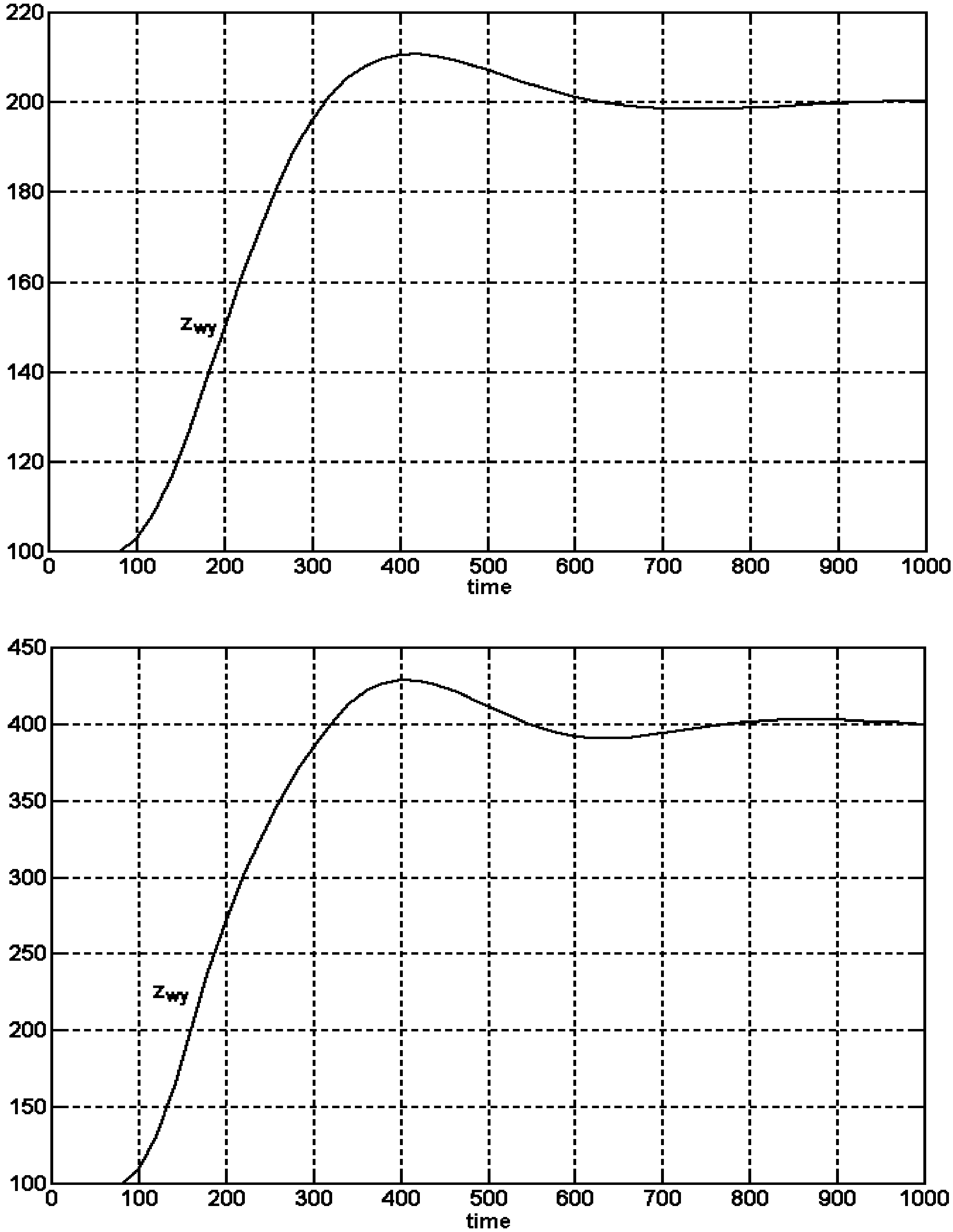


Fig. 8. Responses with fuzzy DMC controller;
set–point change from z_0 = 100 ppm to
a) z_{zad} = 200 ppm; b) z_{zad} = 400 ppm;

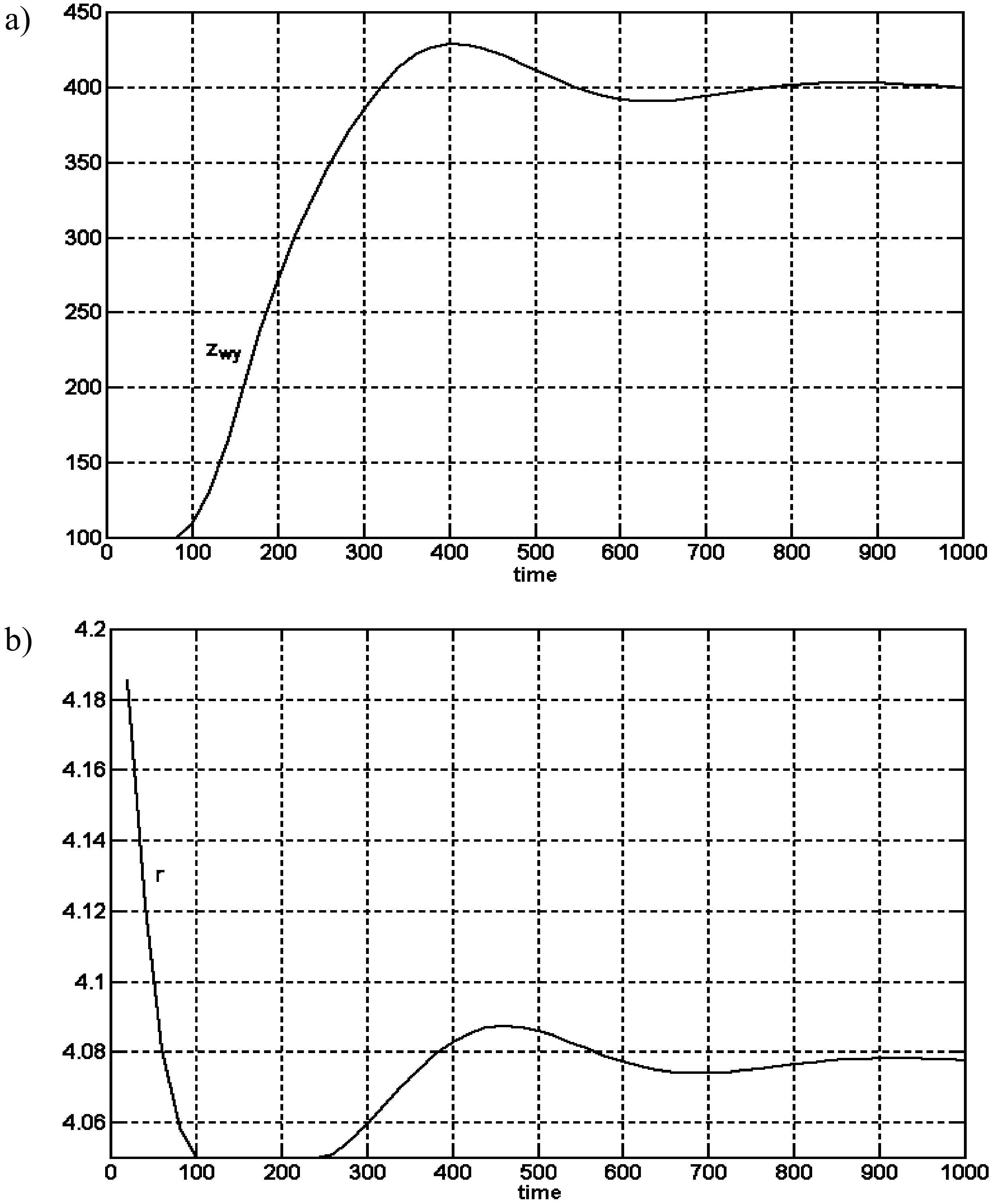


Fig. 9. Responses with fuzzy DMC controller;
set–point change from z_0 = 100 ppm to z_{zad} = 400 ppm;
a) output variable; b) manipulated variable;

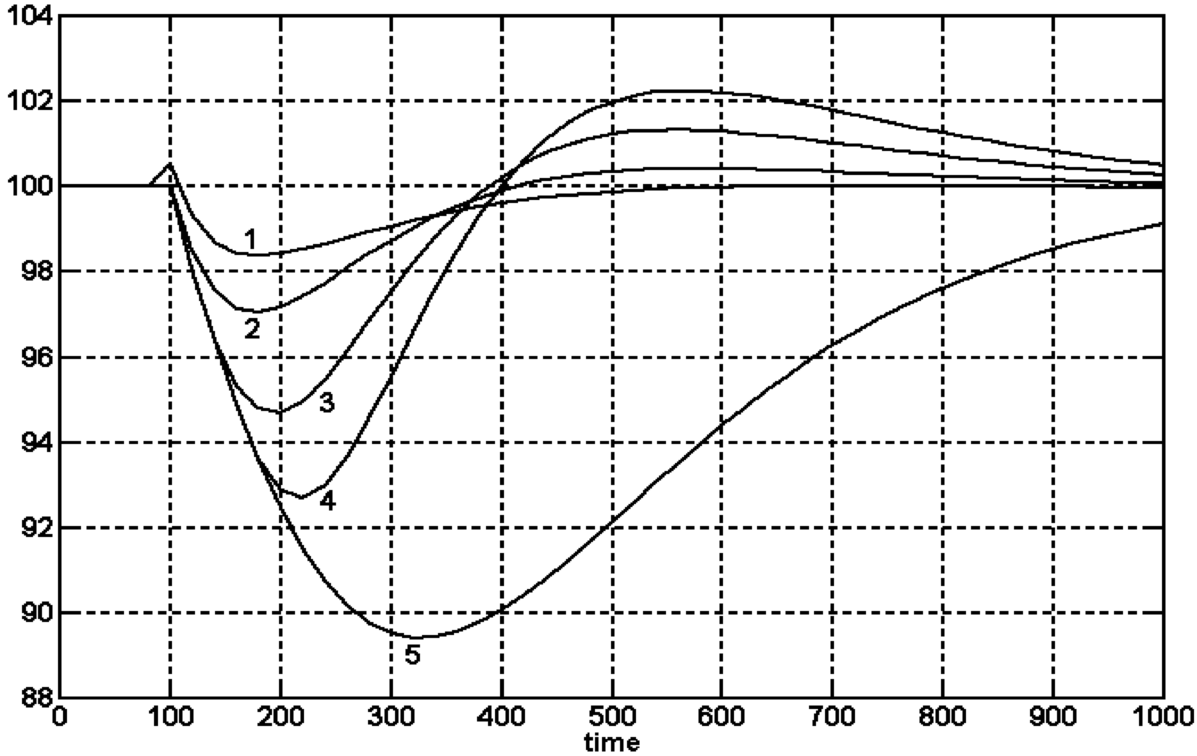


Fig. 10. Responses with fuzzy DMC controller for the step change of measurable disturbance from $x_f = 0{,}81$ to $x_{fb} = 0{,}8$;

1 – one step ahead anticipation of the disturbance change,

2 – immediate measurement,

3 – measurement with delay $2 \cdot T_p$,

4 – measurement with delay $4 \cdot T_p$,

5 – without disturbance measurement.

Tanaka–Sugeno stability criterion

Rule i: if $x(k)$ is $F_1^{\,i}$ and … and $x(k\text{–}n+1)$ is $F_n^{\,i}$, then

$$x^i(k+1) = \boldsymbol{A_i}\cdot\boldsymbol{x}(k),$$

where $\boldsymbol{x}(k) = [x(k), \ldots, x(k\text{–}n+1)]^{\mathrm{T}}$, $\boldsymbol{A}_i = \begin{bmatrix} a_1^i & \ldots & a_{n-1}^i & a_n^i \\ 1 & \ldots & 0 & 0 \\ 0 & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & 0 & 0 \\ 0 & \ldots & 1 & 0 \end{bmatrix}$.

Output of the system:

$$x(k+1) = \frac{\sum_{i=1}^{l} w_i \cdot \boldsymbol{A}_i \cdot \boldsymbol{x}(k)}{\sum_{i=1}^{l} w_i}.$$

Sufficient stability condition for the control system is existence of a positive definite matrix $\boldsymbol{P}$ which satisfies for every local system following inequalities:

$$\boldsymbol{A}_i^T\cdot\boldsymbol{P}\cdot\boldsymbol{A}_i\text{–}\boldsymbol{P}<0 \text{ for } i=1,\ldots, l.$$

Matrix *P* existence conditions

1. Matrix $\boldsymbol{P}$ for given system does not exist if any among local systems is unstable;
2. If $\boldsymbol{A}_i$ (i = 1,…, l) matrices are stable and nonsingular, then $\boldsymbol{A}_i \cdot \boldsymbol{A}_j$ (i = 1,…, l, j = 1,…, l) are stable matrices if matrix $\boldsymbol{P}$ exists.

Methods of looking for matrix *P*

- "Trial and error" method

Positive definite matrix $\boldsymbol{P}_i$ fulfilling i–th inequality should be found;

The following equality can be used:

$\boldsymbol{A}_i^T \cdot \boldsymbol{P} \cdot \boldsymbol{A}_i - \boldsymbol{P} = -\boldsymbol{Q}$, where $\boldsymbol{Q}$ – positive definite matrix.

The next conditions are checked.

If they are not fulfilled, then other $\boldsymbol{Q}$ matrix is chosen and the procedure is repeated.

- Method using LMI

The problem, i.e. $\boldsymbol{P}>0$ i $\boldsymbol{P} - \boldsymbol{A}_i^T \cdot \boldsymbol{P} \cdot \boldsymbol{A}_i > 0$ can be solved using procedures of solving such inequalities (e.g. Matlab LMI toolbox).

Transformation of the examined control system equations

Control plant

Oi: if $y(k)$ is Ya_1^i and … and $y(k–n+1)$ is Ya_n^i and
$u(k)$ is Yb_1^i and … and $u(k–d–m+1)$ is Yb_{m+d}^i then

$$y^i(k+1) = b_1^i \cdot y(k) + \ldots + b_n^i \cdot y(k-n+1) +$$
$$+ c_1^i \cdot u(k-d) + \ldots + c_m^i \cdot u(k-d-m+1)$$

$$y(k+1) = \frac{\sum_{i=1}^{loy} w_i y^i(k+1)}{\sum_{i=1}^{loy} w_i}$$

Controller

Pj: if $y(k–d)$ is Yc_{d+1}^j and … and $y(k–n+1)$ is Yc_n^j and
$u(k–d–1)$ is Yd_2^j and … and $u(k–d–m+1)$ is Yd_m^j then

$$u^j(k-d) = f_2^j \cdot u(k-d-1) + \ldots + f_{m.-d}^j \cdot u(k-d-m+1) +$$
$$+ g_{d+1}^j \cdot e(k-d) + \ldots + g_n^j \cdot e(k-n+1)$$

$$u(k-d) = \frac{\sum_{j=1}^{lou} w_j u^i(k-d)}{\sum_{j=1}^{lou} w_j}$$

Quasi–state vector

$$\boldsymbol{x_b}(k)=[y(k)\ \ldots\ y(k-n+1)\ u(k-d-1)\ \ldots\ u(k-d-m+1)]^{\mathrm{T}}$$

Control plant model

$$\boldsymbol{x_b}(k+1)=\frac{\sum_{i=1}^{loy} w_i \boldsymbol{A^i}}{\sum_{i=1}^{loy} w_i}\cdot \boldsymbol{x_b}(k)+\frac{\sum_{i=1}^{loy} w_i \boldsymbol{B^i}}{\sum_{i=1}^{loy} w_i}\cdot u(k-d),$$

$$y(k+1)=\frac{\sum_{i=1}^{loy} w_i \boldsymbol{C^i}}{\sum_{i=1}^{loy} w_i}\cdot \boldsymbol{x_b}(k+1),$$

$$\boldsymbol{A^i}=\begin{bmatrix} b_1^i & b_2^i & \ldots & b_{n-1}^i & b_n^i & c_2^i & c_3^i & \ldots & c_{m-1}^i & c_m^i \\ 1 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 1 & 0 \end{bmatrix}, \quad \boldsymbol{B^i}=\begin{bmatrix} c_1^i \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$\boldsymbol{C^i}=\begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}.$$

Controller

$$u(k-d) = \frac{\sum_{j=1}^{lou} w_j \boldsymbol{F^j}}{\sum_{j=1}^{lou} w_j} \cdot \boldsymbol{x_b}(k) + \frac{\sum_{j=1}^{lou} w_j \boldsymbol{G^j}}{\sum_{j=1}^{lou} w_j} \cdot \boldsymbol{we}(k),$$

$$\boldsymbol{F^j} = \begin{bmatrix} 0 & 0 & \ldots & 0 & -g_{d+1}^j & \ldots & -g_{n-1}^j & -g_n^j & f_2^j & f_3^j & \ldots & f_{m-1}^j & f_m^j \end{bmatrix}.$$

Control system

$$\boldsymbol{x_b}(k+1) = \frac{\sum_{i=1}^{loy} w_i \cdot \sum_{j=1}^{lou} w_j \cdot \left(\boldsymbol{A^i} + \boldsymbol{B^i} \cdot \boldsymbol{F^j}\right)}{\sum_{i=1}^{loy} w_i \cdot \sum_{j=1}^{lou} w_j} \cdot \boldsymbol{x_b}(k) + $$

$$+ \frac{\sum_{i=1}^{loy} w_i \cdot \sum_{j=1}^{lou} w_j \cdot \boldsymbol{B^i} \cdot \boldsymbol{G^j}}{\sum_{i=1}^{loy} w_i \cdot \sum_{j=1}^{lou} w_j} \cdot \boldsymbol{we}(k),$$

$$y(k+1) = \boldsymbol{C} \cdot \boldsymbol{x_b}(k+1), \ \boldsymbol{C} = \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}.$$

After transformation:

$$x_b(k+1) = \boldsymbol{A} \cdot \boldsymbol{x_b}(k) + \boldsymbol{B} \cdot \boldsymbol{we}(k),$$
$$y(k+1) = \boldsymbol{C} \cdot \boldsymbol{x_b}(k+1),$$

where $\boldsymbol{A} = \dfrac{\sum_{i=1}^{loy} w_i \cdot \sum_{j=1}^{lou} w_j \cdot \boldsymbol{A_{ij}}}{\sum_{i=1}^{loy} w_i \cdot \sum_{j=1}^{lou} w_j}$, $\boldsymbol{B} = \dfrac{\sum_{i=1}^{loy} w_i \cdot \sum_{j=1}^{lou} w_j \cdot \boldsymbol{B^i} \cdot \boldsymbol{G^j}}{\sum_{i=1}^{loy} w_i \cdot \sum_{j=1}^{lou} w_j}$.

The same membership functions in control plant and in controller:

$$\boldsymbol{A} = \frac{\sum_{i=1}^{l} w_i \cdot w_i \cdot \boldsymbol{A_{ii}} + \sum_{i=1}^{l} \sum_{j=i+1}^{l} 2 \cdot w_i \cdot w_j \cdot \boldsymbol{A_{ij*}}}{\sum_{i=1}^{l} \sum_{j=1}^{l} w_i w_j},$$

$$\boldsymbol{A_{ij*}} = \frac{\boldsymbol{A_{ij}} + \boldsymbol{A_{ji}}}{2}.$$

$$
\boldsymbol{A}_{ij} = \begin{bmatrix}
bg_1^{i,j} & bg_2^{i,j} & \cdots & bg_d^{i,j} & bg_{d+1}^{i,j} & \cdots & bg_{n-1}^{i,j} & bg_n^{i,j} & cf_2^{i,j} & cf_3^{i,j} & \cdots & cf_{m-1}^{i,j} & cf_m^{i,j} \\
1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & -g_{d+1}^{j} & \cdots & -g_{n-1}^{j} & -g_n^{j} & f_2^{j} & f_3^{j} & \cdots & f_{m-1}^{j} & f_m^{j} \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & 0
\end{bmatrix},
$$

where $bg_{in}^{i,j} = \begin{cases} b_{in}^{i}; & in < d+1 \\ b_{in}^{i} - c_1^{i} \cdot g_{in}^{j}; & in \geq d+1 \end{cases}$, $cf_{im}^{i,j} = c_{im}^{i} + c_1^{i} \cdot f_{im}^{j}$ $(in = 1, \ldots, n;\ im = 2, \ldots, m)$.

Control system with FDMC controller

Control plant

Oi: if $y(k)$ is $Za_1{}^i$ and … and $y(k–n+1)$ is $Za_n{}^i$ and
$u(k)$ is $Zb_1{}^i$ and … and $u(k–m+1)$ is $Zb_m{}^i$ then

$$y^i(k+1) = b_1{}^i \cdot y(k) +\ldots+ b_n{}^i \cdot y(k-n+1) +$$
$$+ c_1{}^i \cdot u(k) +\ldots+ c_m{}^i \cdot u(k-m+1).$$

Controller

Pj: if $y(k)$ is Zc_1^j and … and $y(k–n+1)$ is Zc_n^j and
$u(k–1)$ is Zd_2^j and … and $u(k–m+1)$ is Zd_m^j then

$$u^j(k) = r_0^j \cdot e(k) + (1+r_1^j) \cdot u(k-1) + (r_2^j - r_1^j) \cdot u(k-2) +\ldots+$$
$$+ (r_{p-1}{}^j - r_{p-2}{}^j) \cdot u(k-(p-1)) - r_{p-1}{}^j \cdot u(k-p).$$

Following coefficients are put into the equation which describes $\boldsymbol{A}_{ij}$ matrices:

$$g_{in}^j = \begin{cases} r_0^j; in = 1 \\ 0; \;\; in > 1 \end{cases}, \qquad f_{im}^j = \begin{cases} 1 + r_1^j; & im = 2 \\ r_{im-1}^j - r_{im-2}^j; & im > 2, im \le p \\ -r_{p-1}^j; & im = p+1 \end{cases},$$

$$bg_{in}^{i,j} = \begin{cases} b_{in}^i - c_1^i \cdot r_0^j; in = 1 \\ b_{in}^i; & in > 1 \end{cases}, \qquad cf_{im}^{i,j} = \begin{cases} c_{im}^i + c_1^i \cdot f_{im}^j; im \ge 2, im \le m \\ c_1^i \cdot f_{im}^j; & im > m \end{cases},$$

$in = 1,\ldots, n,$ $\qquad$ $im = 2,\ldots, p+1.$

$$cf_{im}^{i,j} = \begin{cases} c_{im}^{i} + c_{1}^{i} \cdot (1 + r_{1}^{j}); & im = 2 \\ c_{im}^{i} + c_{1}^{i} \cdot (r_{im-1}^{j} - r_{im-2}^{j}); & im > 2, im \le m \\ c_{1}^{i} \cdot (r_{im-1}^{j} - r_{im-2}^{j}); & im > m, im \le p \\ -c_{1}^{i} \cdot r_{p-1}^{j}; & im = p+1 \end{cases}$$

$$\boldsymbol{A_{ij}} = \begin{bmatrix} b_1^i - c_1^i \cdot r_0^j & b_2^i & \dots & b_{n-1}^i & b_n^i & c_2^i + c_1^i \cdot (1 + r_1^j) & c_3^i + c_1^i \cdot (r_2^j - r_1^j) & \dots & c_m^i + c_1^i \cdot (r_{m-1}^j - r_{m-2}^j) & c_1^i \cdot (r_m^j - r_{m-1}^j) & \dots & c_1^i \cdot (r_{p-1}^j - r_{p-2}^j) & -c_1^i \cdot r_{p-1}^j \\ 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ -r_0^j & 0 & \dots & 0 & 0 & 1 + r_1^j & r_2^j - r_1^j & \dots & r_{m-1}^j - r_{m-2}^j & r_m^j - r_{m-1}^j & \dots & r_{p-1}^j - r_{p-2}^j & -r_{p-1}^j \\ 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 1 & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 1 & 0 \end{bmatrix}$$

Method usage example

$$\text{region 1: } y_{k+1} = 0{,}7 \cdot y_k + 0{,}8 \cdot u_k \,,$$

$$\text{region 2: } y_{k+1} = 0{,}3 \cdot y_k + 0{,}2 \cdot u_k \,,$$

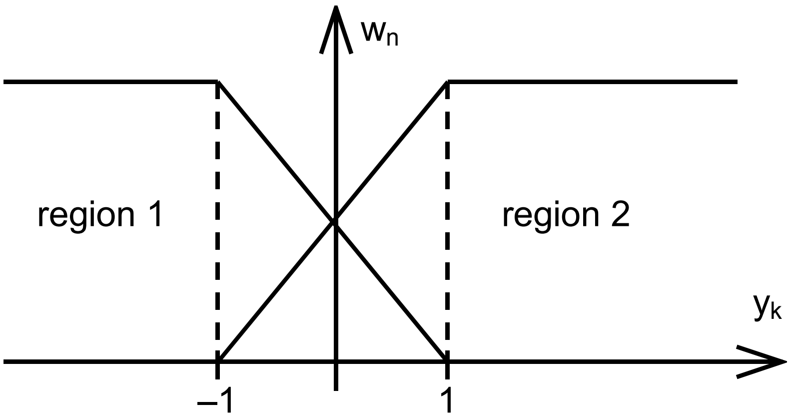


Fig. 11. Membership functions of the fuzzy model

Table 1. Local controllers coefficients

i	0	1	2	3	4
r_i^1	0.9999	–0.6300	–0.4410	–0.3087	–0.2161
r_i^2	2.1857	–0.1567	–0.0470	–0.0141	–0.0042

i	5	6	7	8	9
r_i^1	–0.1512	–0.1060	–0.0741	–0.0519	–0.0362
r_i^2	–0.0013	–0.0004	0	0	0

i	10	11	12	13	14
r_i^1	–0.0254	–0.0179	–0.0124	–0.0088	–0.0061
r_i^2	0	0	0	0	0

i	15	16	17	18
r_i^1	–0.0043	–0.0030	–0.0021	–0.0013
r_i^2	0	0	0	0

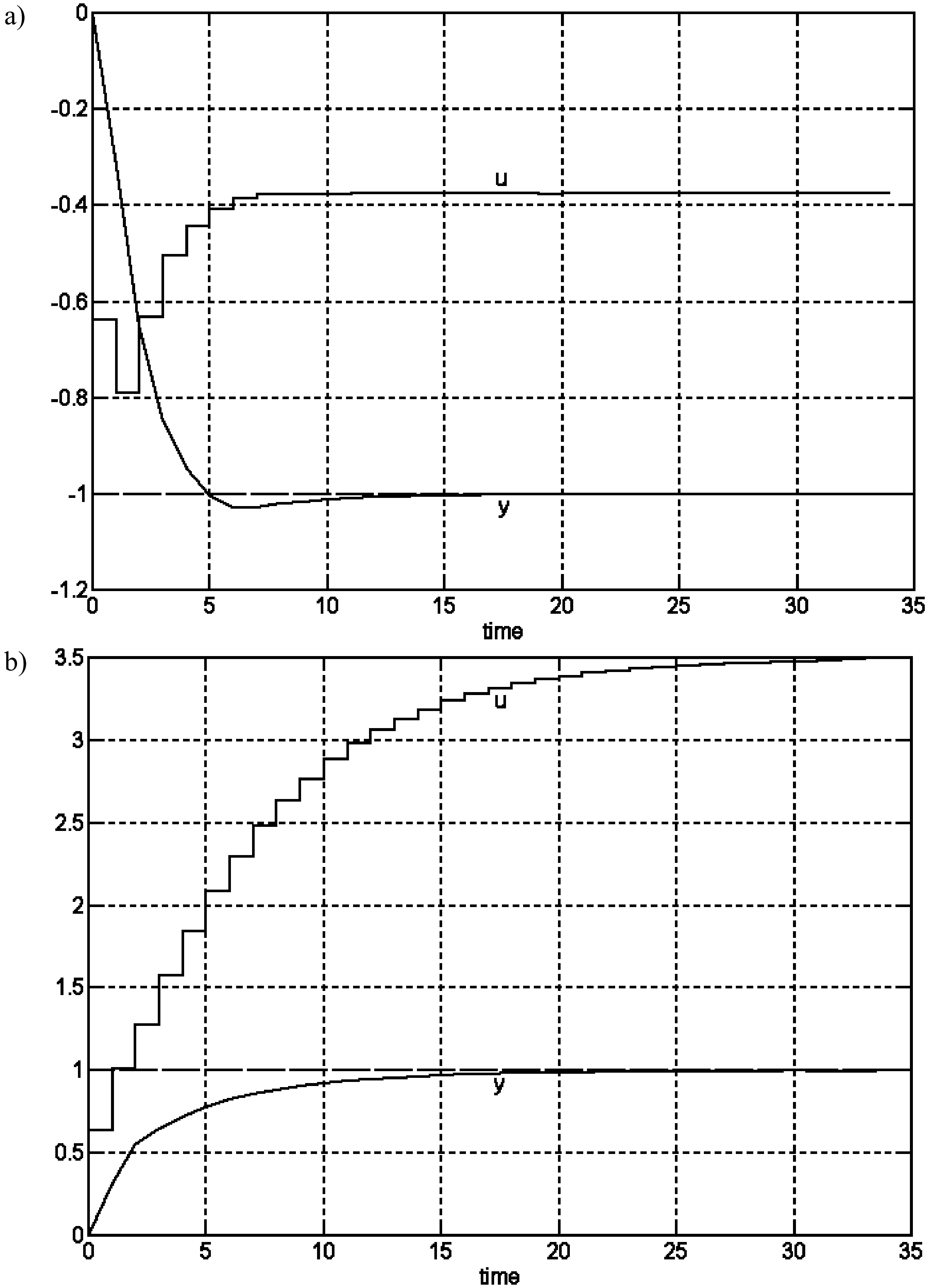


Fig. 12. Responses of the control system with FDMC controller; λ=0,1; set–point change from y_o=0 to a) y_{zad}=–1, b) y_{zad}=1; u – manipulated variable, y – output variable

Table 2. Columns of the A_{11} matrix, i – column number

i=1	i=2	i=3	i=4	i=5	i=6	i=7	i=8	i=9	i=10
–0.0999	0.2960	0.1512	0.1058	0.0741	0.0519	0.0362	0.0255	0.0177	0.0126
–0.9999	0.3700	0.1890	0.1323	0.0926	0.0649	0.0453	0.0319	0.0221	0.0157
0.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	1.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

i=11	i=12	i=13	i=14	i=15	i=16	i=17	i=18	i=19	i=20
0.0086	0.0061	0.0044	0.0029	0.0021	0.0014	0.0010	0.0008	0.0006	0.0010
0.0108	0.0076	0.0055	0.0036	0.0027	0.0018	0.0012	0.0010	0.0008	0.0013
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	1.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000	0.0000

Table 3. Eigenvalues of the A_{11} matrix

–06961 + 01143i	–06961 – 01143i	–06249 + 03334i	–06249 – 03334i	–04835 + 05163i	–04835 – 05163i	–02974 + 06436i	–02974 – 06436i	–00803 + 07076i	–00803 – 07076i
01439 + 06990i	01439 – 06990i	03520 + 06191i	03520 – 06191i	05192 + 04767i	05192 – 04767i	06278 + 02931i	06278 – 02931i	06742 + 00967i	06742 – 00967i

Table 4. Columns of the A_{22} matrix, i – column number

i=1	i=2	i=3	i=4	i=5	i=6	i=7	i=8	i=9	i=10
–0.1371	0.1687	0.0219	0.0066	0.0020	0.0006	0.0002	0.0001	0.0000	0.0000
–2.1857	0.8433	0.1097	0.0329	0.0099	0.0029	0.0009	0.0004	0.0000	0.0000
0.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	1.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

i=11	i=12	i=13	i=14	i=15	i=16	i=17	i=18	i=19	i=20
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	1.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000	0.0000

Table 5. Eigenvalues of the A_{22} matrix

0	0	0	0	0	0	0	0	0	0
0	0	–0.2451 + 0.1259i	–0.2451 – 0.1259i	–0.0415 + 0.2884i	–0.0415 – 0.2884i	0.2410 + 0.2431i	0.2410 – 0.2431i	0.3987 + 0.1212i	0.3987 – 0.1212i

Table 6. Columns of the $\boldsymbol{A}_{12*}$ matrix, i – column number

i=1	i=2	i=3	i=4	i=5	i=6	i=7	i=8	i=9	i=10
–0.4743	0.3743	0.0628	0.0264	0.0132	0.0076	0.0049	0.0034	0.0022	0.0016
–1.5928	0.6067	0.1493	0.0826	0.0512	0.0339	0.0231	0.0162	0.0111	0.0079
0.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	1.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

i=11	i=12	i=13	i=14	i=15	i=16	i=17	i=18	i=19	i=20
0.0011	0.0008	0.0005	0.0004	0.0003	0.0002	0.0001	0.0001	0.0001	0.0001
0.0054	0.0038	0.0027	0.0018	0.0013	0.0009	0.0006	0.0005	0.0004	0.0006
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	1.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000	0.0000

Table 7. Eigenvalues of the $\boldsymbol{A}_{12*}$ matrix

0.8007	0.6343 + 0.2422i	0.6343 – 0.2422i	0.5210 + 0.4372i	0.5210 – 0.4372i	0.3575 + 0.5884i	0.3575 – 0.5884i	0.1612 + 0.6768i	0.1612 – 0.6768i	–0.0472 + 0.6887i
–0.0472 – 0.6887i	–0.2519 + 0.6253i	–0.2519 – 0.6253i	–0.4305 + 0.4994i	–0.4305 – 0.4994i	–0.5656 + 0.3224i	–0.5656 – 0.3224i	–0.6351 + 0.1108i	–0.6351 – 0.1108i	–0.1557

Table 8. Columns of the ***P*** matrix, *i* – column number

i=1	*i*=2	*i*=3	*i*=4	*i*=5	*i*=6	*i*=7	*i*=8	*i*=9	*i*=10
259.5657	–61.7142	–37.5831	–20.3774	–12.3300	–8.1066	–5.5216	–3.7798	–2.6008	–1.7741
–61.7142	57.9341	–6.0267	10.5499	6.5810	4.3054	2.9470	2.0394	1.3972	0.9690
–37.5831	–6.0267	47.5781	–12.2108	6.6402	4.3028	2.8258	1.9284	1.3371	0.9062
–20.3774	10.5499	–12.2108	40.3827	–14.1695	4.5303	2.8790	1.8446	1.2431	0.8579
–12.3300	6.5810	6.6402	–14.1695	36.3146	–14.3572	3.4976	2.1685	1.3539	0.9002
–8.1066	4.3054	4.3028	4.5303	–14.3572	33.2818	–13.8876	2.9102	1.7676	1.0801
–5.5216	2.9470	2.8258	2.8790	3.4976	–13.8876	30.6734	–13.1280	2.5248	1.5087
–3.7798	2.0394	1.9284	1.8446	2.1685	2.9102	–13.1280	28.2631	–12.2304	2.2367
–2.6008	1.3972	1.3371	1.2431	1.3539	1.7676	2.5248	–12.2304	25.9488	–11.2651
–1.7741	0.9690	0.9062	0.8579	0.9002	1.0801	1.5087	2.2367	–11.2651	23.6822
–1.2214	0.6574	0.6322	0.5723	0.6150	0.7073	0.9047	1.3172	1.9954	–10.2669
–0.8217	0.4507	0.4204	0.3989	0.4031	0.4780	0.5840	0.7771	1.1589	1.7775
–0.5575	0.3011	0.2905	0.2607	0.2792	0.3076	0.3894	0.4941	0.6726	1.0170
–0.3863	0.2025	0.1941	0.1783	0.1768	0.2097	0.2449	0.3234	0.4199	0.5794
–0.2592	0.1403	0.1275	0.1178	0.1200	0.1296	0.1643	0.1991	0.2693	0.3541
–0.1845	0.0937	0.0916	0.0751	0.0766	0.0856	0.0980	0.1298	0.1609	0.2207
–0.1241	0.0669	0.0589	0.0542	0.0472	0.0528	0.0627	0.0744	0.1011	0.1271
–0.0767	0.0445	0.0421	0.0337	0.0335	0.0310	0.0366	0.0452	0.0550	0.0757
–0.0408	0.0256	0.0283	0.0230	0.0191	0.0203	0.0199	0.0242	0.0308	0.0382
–0.0315	0.0118	0.0211	0.0168	0.0136	0.0116	0.0122	0.0126	0.0151	0.0195

i=11	*i*=12	*i*=13	*i*=14	*i*=15	*i*=16	*i*=17	*i*=18	*i*=19	*i*=20
–1.2214	–0.8217	–0.5575	–0.3863	–0.2592	–0.1845	–0.1241	–0.0767	–0.0408	–0.0315
0.6574	0.4507	0.3011	0.2025	0.1403	0.0937	0.0669	0.0445	0.0256	0.0118
0.6322	0.4204	0.2905	0.1941	0.1275	0.0916	0.0589	0.0421	0.0283	0.0211
0.5723	0.3989	0.2607	0.1783	0.1178	0.0751	0.0542	0.0337	0.0230	0.0168
0.6150	0.4031	0.2792	0.1768	0.1200	0.0766	0.0472	0.0335	0.0191	0.0136
0.7073	0.4780	0.3076	0.2097	0.1296	0.0856	0.0528	0.0310	0.0203	0.0116
0.9047	0.5840	0.3894	0.2449	0.1643	0.0980	0.0627	0.0366	0.0199	0.0122
1.3172	0.7771	0.4941	0.3234	0.1991	0.1298	0.0744	0.0452	0.0242	0.0126
1.9954	1.1589	0.6726	0.4199	0.2693	0.1609	0.1011	0.0550	0.0308	0.0151
–10.2669	1.7775	1.0170	0.5794	0.3541	0.2207	0.1271	0.0757	0.0382	0.0195
21.4383	–9.2525	1.5710	0.8830	0.4921	0.2924	0.1755	0.0960	0.0527	0.0247
–9.2525	19.2060	–8.2299	1.3702	0.7534	0.4082	0.2337	0.1329	0.0677	0.0337
1.5710	–8.2299	16.9795	–7.2031	1.1729	0.6266	0.3268	0.1775	0.0935	0.0440
0.8830	1.3702	–7.2031	14.7563	–6.1734	0.9781	0.5025	0.2484	0.1254	0.0601
0.4921	0.7534	1.1729	–6.1734	12.5358	–5.1411	0.7862	0.3820	0.1752	0.0811
0.2924	0.4082	0.6266	0.9781	–5.1411	10.3181	–4.1053	0.5991	0.2684	0.1125
0.1755	0.2337	0.3268	0.5025	0.7862	–4.1053	8.1051	–3.0632	0.4213	0.1688
0.0960	0.1329	0.1775	0.2484	0.3820	0.5991	–3.0632	5.9004	–2.0083	0.2626
0.0527	0.0677	0.0935	0.1254	0.1752	0.2684	0.4213	–2.0083	3.7145	–0.9234
0.0247	0.0337	0.0440	0.0601	0.0811	0.1125	0.1688	0.2626	–0.9234	1.5731

Table 9. Eigenvalues of the P matrix

6.7829	8.1576	5.4194	9.7908	4.2718	3.1929	2.1144	12.1812	15.1464	1.1213
18.4993	21.7423	24.6938	28.8905	34.2973	40.7358	48.3791	57.6806	69.9355	285.1187